Strategic Incentives when Implementing Dorfman Testing with Assortative Matching

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Abstract

The Dorfman pooled testing scheme is a process in which individual specimens (e.g., blood, urine, swabs, etc.) are pooled and tested together; if the merged sample tests positive for infection, each specimen from the pool is tested individually. Through this procedure, laboratories can reduce the expected number of tests required to screen a population. The literature has often advocated in favor of using ordered partitions to screen the population, i.e., of pooling subjects with similar probability of infection together, as doing so simultaneously minimizes the expected number of tests, the expected number of false negatives, and the expected number of false positive classifications, provided that certain technical conditions hold. One potential limitation of using ordered partitions, however, is that they may incentivize some subjects to misreport their types to the tester. Indeed, if subjects wish to avoid being detected as infected, ordered partitions would incentivize them to falsely claim that they have a low probability of infection (assuming that pooled testing is subject to dilution effects). These incentives would disappear if subjects were matched randomly, regardless of their probability of infection. In this article, we derive conditions under which ordered partitions outperform matching subjects randomly, despite these incentives.

Keywords: Pooled Testing, Dilution Effects, Heterogenous Priors, Strategic Incentives

1 Introduction

For many infectious diseases, such as HIV, Hepatitis B, and COVID-19, it is a common practice to screen the population through a process known as *Dorfman Screening*. In this process, specimens (e.g., blood, urine, swabs) from different subjects are pooled and tested together; whenever a pooled test detects an infection, each specimen from that group is tested individually. Compared to testing subjects individually, this procedure tends to reduce the overall expected number of tests required to screen a population, as subjects are only tested individually when the pooled test detects an infection.

Several different partitions can be used to form the pools. The literature has shown that *ordered partitions*, i.e., those in which subjects with similar probability of infection are grouped together, satisfy desirable properties. Indeed, Aprahamian, Bish and Bish (2018) and Saraiva (2023) have shown that ordered partitions simultaneously minimize the expected number of tests, the expected number of false negatives, and the expected number of false positives, provided that the dilution effect satisfies certain technical conditions that can be tested empirically.

One potential disadvantage of ordered partitions not considered so far by the literature, however, is that they may give subjects strategic incentives to lie about their prior probability of infection to the tester. Indeed, in the presence of dilution effects, we have that if subjects wish to minimize their probability of receiving a positive result (e.g., because they want to avoid extreme quarantine restrictions), they should falsely claim that they have a low probability of infection; this way, they are matched with other subjects who have a low probability of infection, thus reducing the probability that they receive a positive result in the first stage of testing. In practice, this manipulation can be accomplished by lying about demographic (e.g., address, race, etc.) or behavioral information (e.g., sexual activity, drug usage, etc.). Meanwhile, if subjects were grouped randomly, these strategic incentives would disappear, as subjects' reports would not affect how their samples are matched to form the pools.

This article studies the potential impact that the partitions used to form the pools may have on subjects' incentives to manipulate their perceived probability of infection to influence the pool they get assigned into. In an environment where dishonesty comes with a cost, and all pools are required to have the same size (e.g., because reconfiguring pool sizes is too costly), we show that forming ordered partitions still outperforms matching subjects randomly, in spite of subjects' strategic incentives to lie under the former matching mechanism. This is true if there is a sufficiently large number of subjects who are not sophisticated enough to lie. This result is relevant to situations in which only a fraction of the subjects *play the game* enough

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times to be able to "*play it well*", i.e., taking into account the tester's matching criterion to form the pools (e.g., healthcare workers who are constantly screened for COVID-19). In the online Appendix, we show that the assumption regarding the existence of unsophisticated subjects can be relaxed if subjects are only categorized into two groups: those with high and those with low probability of infection.

2 Environment

Suppose that there is a population $S = \{1, 2, \cdots, N\}$ of subjects to be tested. Each subject can either be infected or not infected. Each subject $i \in S$ is infected with probability $q_i \in [0,1]$.

If a subject is individually tested and she is not infected, the test will classify her as healthy with probability $S_p \in [0,1]$. An infected subject who is individually tested is classified as infected with probability $S_e \in [0,1]$. Using the terminology from clinical trials, S_p corresponds to the *specificity* of the test, while S_e corresponds to its *sensitivity*. We assume that $S_e > 1 - S_p$, so that whenever a test detects the presence of infection, the subject is more likely to be infected compared to the case in which no infection is detected.

In a Dorfman procedure, the subjects to be tested, S, are pooled into disjoint groups, and the samples from subjects belonging to the same group are amalgamated and tested together. If the test detects the presence of infection in the pooled sample, each subject within the pool is tested individually.

Several methods can be used to form the pools. One way is to group subjects randomly, without taking into account their prior probability of infection, into pools of equal size. We designate this type of pooling as *random pooling*. But such a pooling method does not take advantage of the information on subjects' prior probability of infection. An alternative pooling scheme that has often been advocated by the literature is *ordered pooling*. According to this testing scheme, subjects with similar probability of infection are pooled and tested together.

Let $h(I,K)$ be the probability of detecting infection in a pooled sample collected from $K \in \mathbb{N}$ subjects, conditional that exactly $I \in \{0,1,2,\cdots,K\}$ of these subjects are infected. From our definition of sensitivity and specificity, we must have $h(1,1) = Se$ and $h(0,1) = 1 - S_p$. We will refer to h as the *dilution function*. For each $K \in \mathbb{N}$, we assume that $h(I,K)$ is (weakly) increasing in I, i.e., the more infected subjects there are in the pool, the more likely the pooled test will detect infection.

Assumption 1 *$h(I,K)$ is increasing in I.*

3 Optimal partition under truthful reporting

Suppose that all subjects are to be grouped into pools of homogeneous size K and suppose that all subjects report their probability of infection truthfully to the tester. To ensure that all pools have the same size, suppose that the number of subjects, N, is a multiple of K. In this case, the objective of the tester is to form a partition $\Omega = \{G_1, G_2, \cdots, G_{N/K}\}$ of S, with $|G_i| = K$ for all $i \in \{1,2,\cdots,N/K\}$ that minimizes the expected number of tests, the expected number of false positives and the expected number of false negatives. An *ordered partition* corresponds to a partition in which subjects are ranked from lowest to highest probability of infection, and then are matched according to this ranking. So the ordered partition in which all pools have the same size K is given by

$$\{\{1,2,\cdots,K\},\{K+1,K+2,\cdots,2K\},\cdots,\{n-K+1,n-K+2,\cdots,N\}\}.$$

We say the tester implements *ordered pooling* if subjects are matched according to an ordered partition.

For ordered pooling to minimize the expected number of tests and the expected number of false positives, we only need the dilution function $h(I,K)$ to be concave in I (Aprahamian, Bish and Bish (2018) and Saraiva (2023)). This is arguably a very mild assumption, as otherwise the dilution function would be too strong, in which case pooled testing would not be a good alternative to individual testing, as it would generate too many false negatives.

Assumption 2 *(The dilution function is concave) Suppose that $\frac{\partial^2 h(I,K)}{\partial I^2} < 0$ for all $I \geq 0$.*

For ordered pooling to also minimize the expected number of false negatives, a stronger and more technical assumption is required: the dilution function $h(I,K)$ cannot be "too concave" for positive values of I.

Assumption 3 *(The dilution function is not "too concave") Suppose that the dilution function $h(\cdot,K)$ is such that, for all $I \in \{1,2,\cdots,K-1\}$,*

$$\frac{I+1}{2I} h(I+1,k) + \frac{I-1}{2I} h(I-1,k) \geq h(I,k).$$

Proposition 1 *(Saraiva (2023)) If the dilution function $h(I,K)$ satisfies assumptions 1, 2 and 3, ordered pooling minimizes the expected number of tests, the expected number of false positives and the expected number of false negatives.*

As an example, one can easily show that the dilution function

$$h(I,K) = (1-S_p) + (S_p + S_e - 1)\left(\frac{I}{K}\right)^{\delta}, \tag{1}$$

with $\delta \in [0,1]$ is increasing, concave and satisfies hypothesis 1, in which case implementing ordered pooling would be optimal.

4 Subjects' strategic incentives to misreport their likelihood of infection

Let $\hat{Q} = \{\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_m\}$ be the set of possible levels of probability of infection that a subject from S may have. Without loss of generality, suppose that

$$\hat{q}_1 < \hat{q}_2 < \cdots < \hat{q}_m.$$

We will assume that all subjects wish to minimize the probability of receiving a positive result (e.g., to avoid quarantine restrictions). This implies that subjects prefer being matched with others who exhibit low probability of infection. For pedagogic purposes, we will also assume that the dilution function is *strictly* increasing in the number of infected within the pool, which implies that subjects have strict preferences over being matched with others who have low probability of infection.

For each $i \in \{1,2,\cdots,m\}$ there is a set S_i of subjects from S who have probability of infection $\hat{q}_i$ who cannot misreport their probability of infection to the tester, say, because they are too honest or not sophisticated enough to know how their samples will be handled by the tester as a function of their report. For each $i \in \{1,2,\cdots,m\}$ there is also a set S_i^* of subjects who can pay a cost $|i-j|c$ to report that they have probability of infection $\hat{q}_j$, where $c > 0$ is an exogenous constant. So the farther one's report is from the truth, the higher the costs one must incur to sustain their lie.

An ordered partition is used to test subjects, with all pools having size $K \geq 2$. To ensure that all pools have equal size, we assume that the total batch of subjects to be tested is a multiple of K, i.e., we assume that

$$\frac{|S|}{K} = \frac{\sum_i |S_i| + \sum_i |S_i^*|}{K} \in \mathbb{N}.$$

Because this is a static game of complete information, the equilibrium concept used will be the *Nash Equilibrium* (NE). Let $P(\hat{q}_i)$ be the equilibrium probability of someone being infected conditional that one has reported $\hat{q}_i$. We wish to derive conditions under which $P(\hat{q}_i) < P(\hat{q}_{i+1})$ for all $i \in \{1,2,\cdots,m-1\}$. Define $\underline{S} \equiv \min_i\{|S_i|\}$. To prove our result we will require $\underline{S}$ to be larger than the pool size K.

Assumption 4 $\underline{S} > K$.

Assumption 4 ensures that there can be at most $m-1$ groups that end up with a mixed combination of subjects who have made a different report. This implies that, if $\underline{S}$ is sufficiently large, we cannot have an equilibrium in which $P(\hat{q}_j) > P(\hat{q}_{j+1})$. Indeed, if $\underline{S}$ is sufficiently large, we have that, whenever one reports q_i, one is most likely matched with others who have made the same report, so we can "ignore" the unlikely instances in which subjects end up in a group that has a mixed combination of agents who have made different reports. In this case, if we had $P(\hat{q}_j) > P(\hat{q}_{j+1})$, then reporting a low probability of infection would actually increase the probability of being detected as infected, leading some agents to unilaterally deviate towards truthful reporting.

Let

$$\overline{u} \equiv 1 - \hat{q}_1\left[\sum_{I=0}^{K-1} h(I+1,K)\hat{q}_1^I(1-\hat{q}_1)^{K-1-I}S_e\right] - (1-\hat{q}_1)\left[\sum_{I=0}^{K-1} h(I,K)\overline{q}_1^I(1-\hat{q}_1)^{K-1-I}(1-S_p)\right]$$

and

$$\underline{u} \equiv 1 - \hat{q}_m\left[\sum_{I=0}^{K-1} h(I+1,K)\hat{q}_m^I(1-\hat{q}_m)^{K-1-I}S_e\right] - (1-\hat{q}_m)\left[\sum_{I=0}^{K-1} h(I,K)\overline{q}_m^I(1-\hat{q}_m)^{K-1-I}(1-S_p)\right],$$

i.e., $\overline{u}$ is the expected utility that someone with probability of infection $\hat{q}_1$ gets if only matched with subjects in $S_1 \cup S_1^*$, whereas $\underline{u}$ is the expected utility that someone with probability of infection $\hat{q}_m$ gets if only matched with subjects in $S_m \cup S_m^*$.

Lemma 1 *There is a NE in the game characterized by $\left(\hat{Q}, (S_1, S_2, \cdots, S_m), (S_1^*, S_2^*, \cdots, S_m^*), c, K, h(\cdot,K)\right)$. Moreover, any equilibrium from this game must be such that $P(\hat{q}_i) < P(\hat{q}_{i+1})$ for all $i \in \{1,2,\cdots,m-1\}$, if $\frac{K-1}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$.*

Notice that lemma 1 does not require the existence of more "unsophisticated" subjects than "sophisticated" ones (i.e., it does not require $|S_i| > |S_i^*|$), only that the number of "unsophisticated" subjects is sufficiently large in absolute value. This implies that, even in a situation in which there are significantly more sophisticated subjects than unsophisticated ones, subjects' reports may still be informative.

If both the conditions from lemma 1 and proposition 1 hold, we have that, even though ordered pooling may incentivize some subjects to manipulate their perceived probability of infection, this matching criterion still outperforms matching subjects randomly.

Proposition 2 *Suppose that the dilution function $h(\cdot, K)$ satisfies assumptions 1, 2 and 3 and that $\frac{K-1}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$. Then, if the tester implements ordered pooling, matching subjects into homogeneous pools of size K, the resulting NE of the game characterized by $\left(\hat{Q}, (S_1, S_2, \cdots, S_m), (S_1^*, S_2^*, \cdots, S_m^*), c, K, h(\cdot, K)\right)$ generates a lower expected number of tests, a lower expected number of false negatives, and a lower expected number of false positives compared to the case in which the tester matches subjects randomly (into uniform pools of size K).*

Proof: This result follows directly from lemma 1 and from proposition 1. ■

References

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Online Appendix to “Strategic Incentives when Implementing Dorfman Testing with Assortative Matching”

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Contents

A Estimating the dilution function

One can calibrate the parameters of the dilution function from equation (1) by using results from experimental data, as illustrated in the following example.

Example A.1 *Yelin et al. [2020] conduct clinical trials to estimate the probability that a specimen infected with SARS-CoV-2 (the pathogen that causes COVID-19) is correctly detected through a reverse transcription quantitative polymerase chain reaction test (RT-qPCR), under several dilution levels. Table I reports the estimated probability that infection is detected as a function of the dilution level.*[1]

Table I: Sensitivity of RT-qPCR tests extracted from Yelin et al. [2020] under various dilution levels.

I/K	$h(I,K)$
1	0.9623
1/2	0.9623
1/4	0.9623
1/8	0.9208
1/16	0.8472
1/32	0.8472
1/64	0.8094

As studies show that the specificity from RT-PCR tests to detect SARS-CoV-2 are usually close to 100% (e.g., Litchfield et al. [2022]), we set $S_p = 0.99$. Assuming that each observed $h(I,K)$ is given by

$$(1 - S_p) + (S_p + S_e - 1)\left(\frac{I}{K}\right)^{\delta}$$

plus a random idiosyncratic shock $\varepsilon \sim N(0, \sigma^2)$, one can easily estimate S_e and δ through the method of maximum likelihood to obtain $S_e = 0.9890$ and $\delta = 0.0459$. Because $\delta = 0.0459 \in [0,1]$, the calibrated dilution function satisfies assumptions 1, 2 and 3, which, from proposition 1, implies that ordered pooling would simultaneously minimize the expected number of tests, the expected number of false positives and the expected number of false negatives. Figure I depicts the estimated dilution function.

B Strategic Incentives when there are only two possible levels of infection

Suppose that all subjects in S can only be of two types: they either have a high or a low probability of infection. More precisely, we assume that $q_i \in \{q_H, q_L\}$ for all $i \in S$, where $q_H > q_L$. Let $S_H = \{i \in S; q_i = q_H\}$ and $S_H = \{i \in S; q_i = q_L\}$, i.e., S_H is the set of subjects in S with high probability of infection, whereas S_L is the set of subjects in S with low probability of infection.

We will assume that all subjects wish to minimize the probability of receiving a positive result (e.g., to avoid quarantine restrictions). Each subject $i \in S$ can pay an exogenous cost $c > 0$ to misreport his probability of infection. More precisely, each agent can report $\hat{q}_i \in \{q_L, q_H\}$, and if $\hat{q}_i \neq q_i$, then the agent incurs in a cost $c > 0$ from lying.[2] So a subject's final payoff is defined as the probability that the subject is *not* detected as infected conditional on his type and subjects' reports, minus the cost of his report, which is 0 if he told the truth, and $c > 0$ otherwise.

[1] These estimates were extracted from Yelin et al. [2020] assuming a cutoff of 38 polymerase chain reaction (PCR) cycles.

[2] We obtain similar results by assuming that subjects' costs are of private information and are independently drawn by a common distribution $F(\cdot)$. Details are provided in the next section.

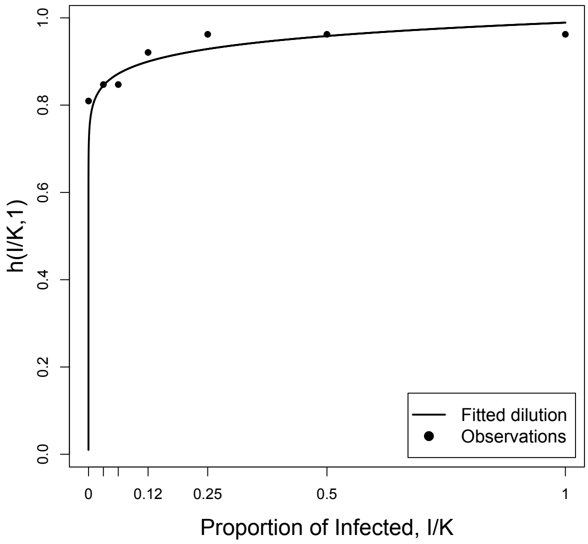

Figure I: Dilution function $h(I,k) = (1 - S_p) + (S_p + S_e - 1)\left(\frac{I}{k}\right)^{1/10}$, with $S_p = .99$, $S_e = 0.989$ and $\delta = 0.0459$. The dots correspond to the sample observations displayed in table I.

Suppose that the tester matches all subjects from S into pools of equal size K according to ordered pooling. Let $\widehat{S}_L$ be the number of subjects in S who claim to have low probability of infection (q_L), and let $\widehat{S}_H \subseteq S_H$ be the set of subjects in S who claim to have a high probability of infection (q_H). We assume that if $|\widehat{S}_L|$ is not a multiple of K, then a random set of subjects is selected to form the group that has a mixed combination of subjects from $\widehat{S}_L$ and $\widehat{S}_H$. More precisely, if $|\widehat{S}_H| \bmod K = x > 0$, then we randomly pick x subjects from $\widehat{S}_H$ and $K - x$ subjects from $\widehat{S}_L$ to form one of the groups, while everyone else is matched with others who have made the same report.

The players from this game are the set of subjects S, who take the tester's matching algorithm (ordered pooling with homogeneous pool sizes) as given. Players simultaneously choose their report $\hat{q}_i \in \{q_L, q_H\}$. The sets S_H and S_L are common knowledge, as well as the cost c. As this corresponds to a static game of complete information, we rely on the *Nash Equilibrium* (NE) concept to predict subjects' behavior.

Lemma B.1 *A NE to the game characterized by* $(S_L, S_H, q_L, q_H, c, K, h(\cdot, K))$ *exists and it must satisfy the following properties:*

a) All subjects in S_L *report their probability of infection truthfully.*

b) (Monotonicity) For any feasible realization of $\widehat{S}_L$*, the probability that a subject is infected conditional that the subject reported* q_L *is less than or equal to the probability that a subject is infected conditional that the subject reported* $\hat{q}_i = q_H$*.*

Proof: The existence of an equilibrium follows directly from the fact that this is a finite game (i.e., there is a finite number of players and they can each take a finite set of actions) of perfect information. So it follows directly from Nash [1951] and Nash [1950], that this game has at least one Nash Equilibrium, possibly in mixed strategies.

Suppose that $S_H \neq \emptyset$ and $S_L \neq \emptyset$ (otherwise the proof would be trivial, as no one would have incentives to misreport their types).

Case 1: Suppose that all subjects in S_L report their probability of infection truthfully, i.e., subjects in S_L report q_L with probability 1.

Let $P(\text{Inf.}|\hat{q}_L, \widehat{S}_L)$ be the probability of a subject being infected, conditional that exactly $|\widehat{S}_L| \leq |S|$ subjects have reported q_L. Similarly, let $P(\text{Inf.}|\hat{q}_H, \widehat{S}_L)$ be the probability of a subject being infected given that she has reported q_H, and given that exactly $|\widehat{S}_L| \leq |S|$ subjects have reported q_L. Because subjects in S_L report their types truthfully with probability 1, we must have

$$P(\text{Inf.}|\hat{q}_H, \widehat{S}_L) = q_H,$$

for any realization of $\widehat{S}_L$.

Moreover, applying the formula for conditional probability, we must also have

$$P(\text{Inf.}|\hat{q}_L, \widehat{S}_L) = \frac{|S_L|q_L + \left(|\widehat{S}_L| - |S_L|\right) q_H}{|\widehat{S}_L|} < q_H,$$

since $|\widehat{S}_L| \geq S_L$ and since $q_L < q_H$.

Case 2: Suppose by way of contradiction that there is a subject in S_L who reports q_H with positive probability. Let $P(\hat{q}_L)$ be the probability of someone being infected conditional that one has reported q_L and let $P(\hat{q}_H)$ be the probability of someone being infected conditional that one reports q_H. Clearly, for a subject in S_L to have incentives to falsely report q_H with positive probability, we must have $P(\hat{q}_H) < P(\hat{q}_L)$. But if $P(\hat{q}_H) < P(\hat{q}_L)$, then subjects in S_H would have incentives to report q_H with probability 1, a contradiction with $P(\hat{q}_H) < P(\hat{q}_L)$.$\rightarrow\leftarrow$

■

Part a) from lemma B.1 follows from the fact that subjects in S_L clearly have no incentives to claim that they have a high probability of infection. Indeed, in our model we are assuming that agents are trying to avoid being detected as infected (say, to avoid quarantine restrictions). So if someone in S_L falsely claimed to have a high probability of infection, this person would be matched with subjects with high probability of infection, thus *increasing* the probability that this subject was classified as infected, which would clearly go against this subject's interests.

Part b) from Lemma B.1 states that, even though ordered pooling causes some subjects in S_H to misreport their true types, the signal received by the tester is still informative (though not as informative as it would be if there was no manipulation). From proposition 1 this implies that, provided that the dilution function satisfies assumptions 1, 2 and 3, implementing ordered pooling still outperforms implementing random pooling, the main result from this section.

Proposition B.2 *Suppose that the dilution function $h(\cdot, K)$ is concave and satisfies assumptions 1, 2 and 3. If the tester implements ordered pooling, the resulting NE of the game characterized by $(S_L, S_H, q_L, q_H, c, K, h(\cdot, K))$ generates a lower expected number of tests, a lower expected number of false negatives and a lower expected number of false positives compared to the case in which the tester matches subjects randomly (into uniform pools of size K).*

Proof: This result follows directly from part b) of lemma B.1 and from proposition 1. ■

Example B.1 *According to the CDC, between September 18 and December 24 from 2022, the probability of SARS-CoV-2 infection from unvaccinated individuals aged 12 or above was approximately 2.8 times the probability of infection from those who had received bivalent doses (Centers for Disease Control and Prevention [2023]). So let us assume that half of the population is not vaccinated while the other half is fully vaccinated with bivalent doses, and that the overall prevalence rate of COVID-19 among subjects who are tested is 10%. Suppose that a fraction $p \in [0,1]$ of unvaccinated individuals falsely claim that they are vaccinated. Plot II depicts the percentage reduction in the expected number of tests, expected number of false negatives and expected number of false positives when implementing ordered pooling as opposed to random pooling for different pool sizes and different values of p, using the dilution function estimated in example A.1. In these plots we assume the batch size of subjects to be tested to be equal to 10,000. Under random pooling, whenever the batch size was not a multiple of the pool size K, a group of subjects was randomly selected to form the smaller pool (if the smaller group was comprised of a single subject, this subject was individually tested, with no followup tests). Under ordered pooling, whenever this occurred, a group of subjects who reported to have high probability of infection was assigned into the smaller group.*[3]

As it can be seen in the figure, the benefits of implementing ordered pooling as opposed to random pooling can be quite substantial, even if a considerable fraction of the population misreport their type.

B.1 Incomplete information specification

Suppose that all subjects in S can only be of two types: they either have a high or a low probability of infection. More precisely, we assume that $q_i \in \{q_H, q_L\}$ for all $i \in S$, where $q_H > q_L$. Let $S_H = \{i \in S; q_i = q_H\}$ and $S_H = \{i \in S; q_i = q_L\}$, i.e., S_H is the set of subjects in S with a high probability of infection, whereas S_L is the set of subjects in S with a low probability of infection.

Suppose that all subjects wish to minimize the probability of receiving a positive result (e.g., to avoid quarantine restrictions). Each subject $i \in S$ can pay a cost $c_i \geq 0$ to misreport her probability of infection. More precisely, each agent can report $\hat{q}_i \in \{q_L, q_H\}$, and if $\hat{q}_i \neq q_i$, then the agent incurs in a cost $c_i \geq 0$ from lying. The costs are

[3] The literature has conjectured that it is usually optimal to have those with smaller probability of infection to be assigned into smaller groups (e.g., McMahan, Tebbs and Bilder [2012]). Though Aprahamian, Bish and Bish [2019] have found a counter-example to this rule of thumb, they have also shown analytically that, in the absence of dilution effects, it is always optimal to have those with highest probability of infection to be the ones, if any, that should be individually tested. At any rate, because we are using a large batch size, the criterion used to determine who should be allocated into the smaller pool have little impact in our final results.

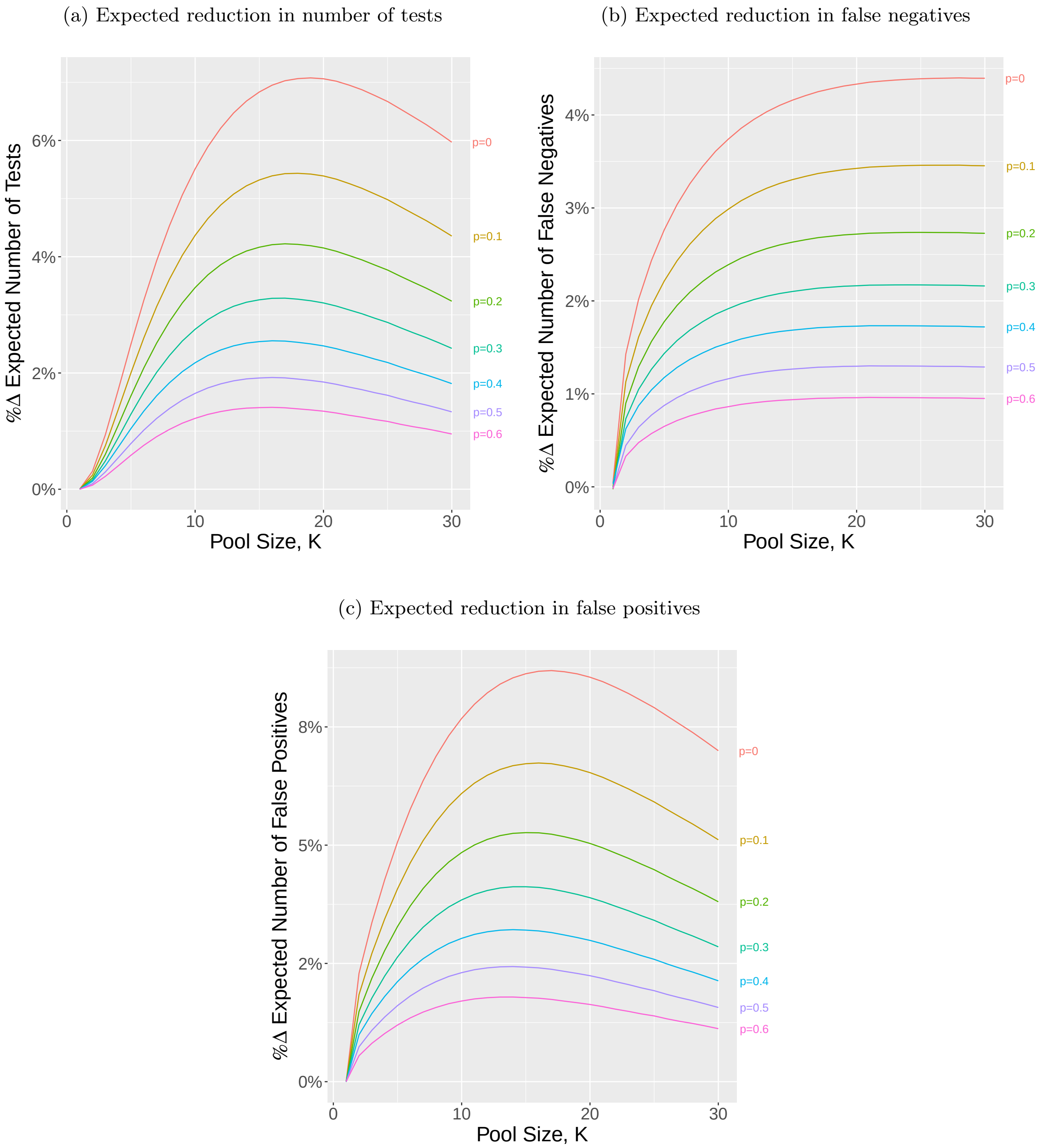

Figure II: Percentage savings in the expected number of tests, the expected number of false negatives and the expected number of false positives when implementing ordered pooling as opposed to random pooling as a function of $p \in [0, 1]$, the probability that someone with high probability of infection falsely claims that he has low probability of infection, for different pool sizes K. For this simulation, we use the dilution function estimated in example A.1, and assume that half of the population is unvaccinated and therefore has a high probability of infection given by 0.15, while the other half is fully vaccinated and therefore has a low probability of infection, given by 0.05.

independently drawn from the same continuous distribution with cdf $F(\cdot)$ and pdf $f(\cdot) = F'(\cdot)$. We assume costs to be subjects' private information, i.e., each subject knows their own cost of lying, but not the costs from other subjects.

Suppose that the tester matches all subjects S into pools of equal size K according to ordered pooling. Let $\widehat{S_L}$ be the set of subjects in S who claim to have a low probability of infection (q_L), and let $\widehat{S_H} \subseteq S_H$ be the set of subjects in S who claim to have a high probability of infection (q_H). We assume that if $\widehat{S_L}$ is not a multiple of K, then a random set of subjects is selected to form the group that has a mixed combination of subjects from $\widehat{S_L}$ and $\widehat{S_H}$. More precisely, if $S_H \bmod K = x > 0$, then we randomly pick x subjects from $\widehat{S_H}$ and $k - x$ subjects from $\widehat{S_L}$ to form one of the groups, while everyone else is matched with others who have made the same report.

The players from this game are the set of subjects S, who take the tester's matching algorithm (ordered pooling with homogeneous pool sizes) as given. A strategy from a player is a function that maps the player's private cost c_i into a report $\hat{q}_i \in \{q_L, q_H\}$. The sets S_H and S_L are common knowledge, as well as the cdf $F(\cdot)$ governing the distribution of $(c_i)_{i\in S}$. As this corresponds to a static game of incomplete information, we rely on the *Bayesian Nash Equilibrium* (BNE) concept to predict subjects' behavior. According to this equilibrium concept each player adopts a strategy that maximizes their expected payoff, given everyone else's strategies, and given that agents correctly form their expectations by implementing the *Bayes' Rule*.

Lemma B.3 *There exists a BNE to the game characterized by* $(S_L, S_H, q_L, q_H, F(\cdot), K, h(\cdot, K))$ *that satisfies the following properties:*

a) Each subject i in S_L reports her probability of infection truthfully regardless of her realization of c_i.

b) (Monotonicity) The probability that a subject is infected conditional that the subject reported $\hat{q}_i = q_L$ is strictly lower than the probability of infection conditional that the subject reported $\hat{q}_i = q_H$.

Proof: Suppose that $S_H \neq \emptyset$ and $S_L \neq \emptyset$ (otherwise the proof would be trivial, as no one would have incentives to misreport their types). Suppose that all subjects in S_L report their probabilities of infection truthfully regardless of their cost c_i (i.e., suppose that $\hat{q}_i = q_L$ for all $i \in S_L$).

Suppose that all subjects in S_H adopt the same strategy $\hat{q}_i$ mapping c_i into $\{q_L, q_H\}$. Then clearly, if a subject $i \in S_H$ with cost c_i has incentives to choose $\hat{q}_i = q_L$ (i.e., to lie) then a subject $j \in S_H$ with cost $c_j < c_i$ also has incentives to choose $\hat{q}_j = q_L$. So given the truthful reporting strategy from subjects in S_L, there is a cutoff point $\bar{c}$ such that a subject i in S_H falsely reports $\hat{q}_i = q_L$ if and only if $c_i \leq \bar{c}$.

Given subjects' strategies, let $P(\text{Inf.}|\hat{q}_L)$ be the probability that a subject is infected conditional that she has reported q_L, and let $P(\text{Inf.}|\hat{q}_H)$ be the probability that a subject is infected conditional that she has reported q_H. Then, by applying the formula for conditional probability we have that

$$\begin{aligned}
P(\text{Inf.}|\hat{q}_L) &= \frac{P(\text{Inf.} \cap \hat{q}_L)}{P_i(\hat{q}_L)} \\
&= \frac{P\left(\text{Inf.} \cap [(\hat{q}_L \cap q_L) \cup (\hat{q}_L \cap q_H)]\right)}{P((\hat{q}_L \cap q_L) \cup (\hat{q}_L \cap q_H))} \\
&= \frac{P\left([\text{Inf.} \cap (\hat{q}_L \cap q_L)] \cup [I \cap (\hat{q}_L \cap q_H)]\right)}{P(\hat{q}_L \cap q_L) + P(\hat{q}_L \cap q_H)} \\
&= \frac{P\left(\text{Inf.} \cap (\hat{q}_L \cap q_L)\right) + P\left(\text{Inf.} \cap (\hat{q}_L \cap q_H)\right)}{P(q_L) + P(q_H)P(\hat{q}_L|q_H)} \\
&= \frac{P(q_L)P\left(\text{Inf.}|q_L\right) + P(q_H)P\left(\hat{q}_L|q_H\right)P(\text{Inf.}|q_H)}{\frac{|S_L|}{|S|} + \frac{|S_H|}{|S|}P(c_i \leq \bar{c})} \\
&= \frac{\frac{|S_L|}{|S|}q_L + \frac{|S_H|}{|S|}Prob(c_i \leq \bar{c})q_H}{\frac{|S_L|}{|S|} + \frac{|S_H|}{|S|}Prob(c_i \leq \bar{c})} = \frac{|S_L|q_L + |S_H|Prob(c_i \leq \bar{c})q_H}{|S_L| + |S_H|Prob(c_i \leq \bar{c})} \qquad (1)
\end{aligned}$$

Now let us compute the probability that a subject is infected conditional that she reports q_H.

Case 1: Suppose that $P(\hat{q}_H|q_H) = Prob(c_i > \bar{c}) > 0$. Then, by applying the formula of conditional probability we have that

$$\begin{aligned}
P(\text{Inf.}|\hat{q}_L) &= \frac{P(\text{Inf.} \cap \hat{q}_H)}{P(\hat{q}_H)} \\
&= \frac{P(\text{Inf.} \cap (q_H \cap \hat{q}_H))}{P(q_H \cap \hat{q}_H)} \\
&= \frac{P(q_H)P(\hat{q}_H|q_H)P(\text{Inf.}|q_H)}{P(q_H)P(\hat{q}_H|q_H)} \\
&= \frac{S_H Prob(c_i > \bar{c})q_H}{S_H Prob(c_i > \bar{c})} \\
&= q_H. \qquad (2)
\end{aligned}$$

Because $q_L < q_H$, equations 1 and 2 imply that

$$P(\text{Inf.}|\hat{q}_L) < q_H = P(\text{Inf.}|\hat{q}_H).$$

Therefore, despite the possibility that subjects in S_H may manipulate their signals, subjects' signals are still informative, given that they adopt this strategy profile. So subjects prefer being matched with others who report $\hat{q}_i = \hat{q}_L$ (i.e., those in $\hat{S}_L$) as opposed to being matched with subjects who report $\hat{q}_i = q_H$ (i.e., those in $\hat{S}_H$).

Now it only remains to show that subjects in S_L have no incentives to deviate from truthful reporting. Disregarding one's costs of misreporting one's type (or equivalently, assuming $c_i = 0$), let u_L^L be the expected utility that someone in S_L gets from being matched only with other subjects in $\hat{S}_L^L$, let u_H^L be the expected utility that someone in S_L gets from being matched only with subjects in $\hat{S}_H$, and let u_M^L be the expected utility that someone in S_L gets from being matched to the group that has a combination of subjects from $\hat{S}_L$ and $\hat{S}_H$. Because subjects prefer being matched with subjects in $\hat{S}_L$, we have that

$$u_L^L \leq u_M^L \leq u_H^L.$$

Let p_L be the probability that a subject i is matched only with subjects in $\hat{S}_L$ given that the subject reports $\hat{q}_i = q_L$. Then, the expected payoff that a subject from S_L gets from reporting $\hat{q}_i = q_L$ is given by

$$p_L u_L^L + (1 - p_L) u_M^L \geq u_M^L. \tag{3}$$

Let p_H be the probability that a subject i is matched only with subjects in $\hat{S}_H$ given that the subject reports $\hat{q}_i = q_H$. Then, the expected payoff that a subject from S_L gets from reporting $\hat{q}_i = q_H$ is given by

$$p_H u_H^L + (1 - p_H) u_M^L \geq u_M^L. \tag{4}$$

From inequalities 3 and 4, we have that

$$p_L u_L^L + (1 - p_L) u_M^L \geq p_H u_H^L + (1 - p_H) u_M^L.$$

Therefore, even if the costs of misreporting one's type was zero, subjects in S_L should have no incentives to deviate from truthful reporting.

Case 2: Suppose that $P_i(\hat{q}_H | q_H) = Prob(c_i > \overline{c}) = 0$. In this case, all subjects in S report low probability of infection, in which case the expected assignment from a subject in S_L would be the same regardless of her report. Because misreporting one's type is costly, no subject in S_L would have incentives to *unilaterally* deviate from truthful reporting.[4]

■

Proposition B.4 *Suppose that the dilution function $h(\cdot, K)$ is concave and satisfies assumptions 1, 2 and 3. If the tester implements ordered pooling matching subjects into homogeneous pools of size K, there is a BNE to the game characterized by $(S_L, S_H, q_L, q_H, F(\cdot), K, h(\cdot, K))$ that generates a lower expected number of tests, a lower expected number of false negatives and a lower expected number of false positives compared to the case in which the tester matches subjects randomly (into uniform pools of size K) ignoring subjects' reported probability of infection.*

Proof: This result follows directly from part b) from lemma B.3 and from proposition 1. ■

C Three or more possible levels of infection

C.1 Proof or proposition 1

Let us introduce some notation: given agents' strategies, let $u^l_{\{i\}}$ be the probability that a subject of type $\hat{q}_l$ is *not* detected as infected conditional that the subject was assigned to a group in which all remaining subjects have *reported* to have probability of infection $\hat{q}_i$. Similarly, let $u^l_{\{i,j\}}$ be the probability that a subject of type $\hat{q}_l$ is not detected as infected conditional that he was matched into a group in which a fraction of the subjects have reported $\hat{q}_i$ while the other fraction reported $\hat{q}_j$. Because the probability that one is detected as infected is increasing in one's own probability of infection, and also in the probability of infection from the remaining subjects within the group, we must have that

$$\overline{u} \equiv 1 - \hat{q}_1 \left[\sum_{I=0}^{K-1} h(I+1, K) \hat{q}_1^I (1 - \hat{q}_1)^{K-1-I} S_e \right] - (1 - \hat{q}_1) \left[\sum_{I=0}^{K-1} h(I, K) \overline{q}_1^I (1 - \hat{q}_1)^{K-1-I} (1 - S_p) \right].$$

[4]Notice that, in accordance with the BNE concept, we only consider unilateral deviations, i.e., treating the strategies adopted by everyone else as constant. Also notice that multilateral deviations (i.e., blocking coalitions) would not be adequate to our environment, as subjects probably cannot communicate with one another, let alone sign a binding contract to form a blocking coalition.

corresponds to an upper bound to $u^l_{\{i\}}$ and $u^l_{\{i,j\}}$, whereas

$$\underline{u} \equiv 1 - \hat{q}_m \left[\sum_{I=0}^{K-1} h(I+1,K)\hat{q}_m^I (1-\hat{q}_m)^{K-1-I} S_e \right] - (1-\hat{q}_m) \left[\sum_{I=0}^{K-1} h(I,K)\overline{q}_m^I (1-\hat{q}_m)^{K-1-I} (1-S_p) \right].$$

corresponds to a lower bound. In words, $\overline{u}$ is the utility that someone with probability of infection $\hat{q}_1$ gets if only matched with subjects in $S_1 \cup S_1^*$. Similarly, $\underline{u}$ is the utility that someone with probability of infection $\hat{q}_m$ gets if only matched with subjects in $S_m \cup S_m^*$.

Let $p^i_{\{i\}}$ be the probability that a subject who reports $\hat{q}_i$ ends up matched to a group in which everyone else also has reported $\hat{q}_i$. Similarly, let $p^i_{\{i,j\}}$ be the probability that a subject who reports $\hat{q}_i$ ends up matched to a group that has a mixed combination of subjects who have reported probability of infection $\hat{q}_i$ and others who have reported probability of infection $\hat{q}_j$.

It follows directly from assumption 4 that $p^i_{\{i,j\}} = 0$ whenever $|i-j| > 1$. Therefore, we have that the expected utility that a subject with probability of infection $\hat{q}_l$ gets from reporting probability of infection $\hat{q}_1$ is given by

$$p^1_{\{1\}} u^l_{\{1\}} + p^1_{\{1,2\}} u^l_{\{1,2\}} - (l-1)c,$$

where $p^1_{\{1\}} + p^1_{\{1,2\}} = 1$. Similarly, the expected utility that a subject with probability of infection $\hat{q}_l$ gets from reporting a probability of infection equal to $\hat{q}_i$, with $i \in \{2,3,\cdots,m-1\}$ is given by

$$p^i_{\{i\}} u^l_{\{i\}} + p^i_{\{i,i+1\}} u^l_{\{i,i+1\}} + p^i_{\{i-1,i\}} u^l_{\{i-1,i\}} - |l-i|c,$$

where $p^i_{\{i\}} + p^i_{\{i,i+1\}} + p^i_{\{i-1,i\}} = 1$.

Finally, the expected utility that a subject with probability of infection $\hat{q}_l$ gets from reporting a probability of infection equal to $\hat{q}_m$ is given by

$$p^m_{\{m\}} u^l_{\{m\}} + p^i_{\{m-1,m\}} u^l_{\{m-1,m\}} - (m-l)c,$$

where $p^m_{\{m\}} + p^m_{\{m-1,m\}} = 1$.

One can show that $p^i_{\{i,j\}} \leq \frac{K-1}{|S_i|}$. Indeed, the maximum number of subjects who can end up in a group that has a mixed combination of subjects who has reported $\hat{q}_i$ and $\hat{q}_j$ is given by $K-1$. Because subjects are assigned into this "*mixed group*" through a fair lottery, the probability that they end up assigned to this group in the most conservative scenario is given by $(K-1)/|S_i|$.

Remark C.1 *Assumption 4 implies that:*

a) $p^i_{\{i,j\}} = 0$ *whenever* $|i-j| > 1$, *and*

b) $p^i_{\{i,j\}} \leq \frac{K-1}{\underline{S}}$ *whenever* $|i-j| = 1$.

c) If $i = 1$, *then* $p^i_{\{i\}} + p^i_{\{i,i+1\}} = 1$. *If* $i \in \{2,3,\cdots,m-1\}$ *then* $p^i_{\{i\}} + p^i_{\{i,i+1\}} + p^i_{\{i,i-1\}} = 1$.

To prove our result, we first show that we must have $P(\hat{q}_1) < P(\hat{q}_2)$. Then we apply induction to show that we must have $P(\hat{q}_i) < P(\hat{q}_{i+1})$ for all $i \in \{2,3,\cdots,m-2\}$. Finally, we show that $P(\hat{q}_{m-1}) < P(\hat{q}_m)$ (a result that does not rely on induction).

- **Existence:** The existence of an equilibrium follows directly from the fact that this is a finite game (i.e., there is a finite number of players and they can each take a finite set of actions) of perfect information. So it follows directly from Nash [1951] and Nash [1950], that this game has at least one Nash Equilibrium, possibly in mixed strategies.

- $\mathbf{P(\hat{q}_i) < P(\hat{q}_{i+1})}$:

  1. $P(\hat{q}_1) < P(\hat{q}_2)$.
     Suppose by way of contradiction that there is a NE in which $P(\hat{q}_1) \geq P(\hat{q}_2)$. Then there must be some $l > 2$, such that subjects in S_l^* report $\hat{q}_1$ with positive probability. For this to be true, the expected payoff from subjects in S_l^* of reporting $\hat{q}_1$ must be greater than or equal the expected payoff they would get from reporting $\hat{q}_2$. So we must have

     $$p^1_{\{1\}} u^l_{\{1\}} + p^1_{\{1,2\}} u^l_{\{1,2\}} - (l-1)c \geq p^2_{\{2\}} u^l_2 + p^2_{\{1,2\}} u^l_{\{1,2\}} + p^2_{\{2,3\}} u^l_{\{2,3\}} - (l-2)c \quad (5)$$

     Because we are assuming that those who report $\hat{q}_1$ are more likely to be infected than those who report $\hat{q}_2$, we must have

     $$u^l_{\{1\}} < u^l_{\{1,2\}} < u^l_{\{2\}}.$$

Therefore, inequality (5) implies that

$$\begin{aligned}
&u^l_{\{1,2\}} \geq (1-p^2_{\{2,3\}})u^l_{\{1,2\}} + p^2_{\{2,3\}}u^l_{\{2,3\}} + c\\
\Rightarrow &p^2_{\{2,3\}}(u^l_{\{1,2\}} - u^l_{\{2,3\}}) \geq c\\
\Rightarrow &\frac{(K-1)}{\underline{S}}(\overline{u}-\underline{u}) \geq c\\
\Rightarrow &\frac{(K-1)}{\underline{S}} \geq \frac{c}{(\overline{u}-\underline{u})},
\end{aligned}$$

a contradiction with the hypothesis that $\frac{(K-1)}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$.

2. $P(\hat{q}_2) < P(\hat{q}_3)$.
Suppose by way of contradiction that there is a NE in which $P(\hat{q}_2) \geq P(\hat{q}_3)$. Then, either there is some $l > 3$, with subjects in S^*_l reporting $\hat{q}_2$ with positive probability, or there are some subjects in S^*_1 reporting $\hat{q}_3$ with positive probability.
   (a) Suppose that there is an $l > 3$ such that (some) subjects in S^*_l report $\hat{q}_2$ with positive probability. Then, the expected payoff that subjects from S^*_l get from reporting $\hat{q}_2$ must be greater than or equal to their expected payoff from reporting $\hat{q}_3$:

$$p^2_{\{2\}}u^l_{\{2\}} + p^2_{\{1,2\}}u^l_{\{1,2\}} + p^2_{\{2,3\}}u^l_{\{2,3\}} - (l-2)c \geq p^3_{\{3\}}u^l_{\{3\}} + p^3_{\{2,3\}}u^l_{\{2,3\}} + p^3_{\{3,4\}}u^l_{\{3,4\}} - (l-3)c. \quad (6)$$

   But from part 1 we already know that we must have

$$u^l_{\{1,2\}} > u^l_{\{2\}}. \quad (7)$$

   Moreover, because we are assuming (by way of contradiction) that $P(\hat{q}_3) < P(\hat{q}_2)$, we must have

$$u^l_{\{2\}} < u^l_{\{2,3\}} < u^l_{\{3\}}. \quad (8)$$

   Inequalities (6), (7) and (8) then imply that

$$\begin{aligned}
&\left(1-p^2_{\{1,2\}}\right)u^l_{\{2,3\}} + p^2_{\{1,2\}}u^l_{\{1,2\}} \geq \left(1-p^3_{\{3,4\}}\right)u^l_{\{2,3\}} + p^3_{\{3,4\}}u^l_{\{3,4\}} + c\\
\Rightarrow &\left(1-\frac{K-1}{\underline{S}}\right)u^l_{\{2,3\}} + \frac{K-1}{\underline{S}}u^l_{\{1,2\}} \geq \left(1-\frac{K-1}{\underline{S}}\right)u^l_{\{2,3\}} + \frac{K-1}{\underline{S}}u^l_{\{3,4\}} + c\\
\Rightarrow &\frac{K-1}{\underline{S}}(\overline{u}-\underline{u}) \geq c,
\end{aligned}$$

   a contradiction with the hypothesis that $\frac{(K-1)}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$.
   (b) Suppose that (some) subjects in S^*_1 report $\hat{q}_3$ with positive probability. Then, the expected payoff that subjects from S^*_1 get from reporting $\hat{q}_3$ must be greater than or equal to their expected payoff from reporting $\hat{q}_1$.
      – **Case 1:** Suppose that

$$u^2_{\{3\}} > u^1_{\{1\}}.$$

      Then, there must be an $l > 3$ such that subjects from S^*_l report $\hat{q}_1$ with positive probability. Then, we must have

$$p^1_{\{1\}}u^l_{\{1\}} + p^1_{\{1,2\}}u^l_{\{1,2\}} - (l-1)c \geq p^3_{\{3\}}u^l_{\{3\}} + p^3_{\{2,3\}}u^l_{\{2,3\}} + p^3_{\{3,4\}}u^l_{\{3,4\}} - (l-3)c \quad (9)$$

      From part 1 we already know that

$$u^l_{\{1\}} > u^l_{\{1,2\}}.$$

      Therefore, inequality (9) implies that

$$\begin{aligned}
&u^l_{\{1\}} \geq p^3_{\{3\}}u^l_{\{3\}} + p^3_{\{2,3\}}\underline{u} + p^3_{\{3,4\}}\underline{u} + 2c\\
\Rightarrow &u^l_{\{1\}} \geq \left(1-\frac{2(K-1)}{\underline{S}}\right)u^l_{\{3\}} + \frac{(K-1)}{\underline{S}}\underline{u} + \frac{(K-1)}{\underline{S}}\underline{u} + 2c\\
\Rightarrow &\underbrace{(u^l_{\{1\}} - u^l_{\{3\}})}_{<0} + \frac{2(K-1)}{\underline{S}}\underbrace{u^l_{\{3\}}}_{\leq\overline{u}} - \frac{2(K-1)}{\underline{S}}\underline{u} \geq 2c\\
\Rightarrow &\frac{2(K-1)}{\underline{S}}(\overline{u}-\underline{u}) \geq 2c,
\end{aligned}$$

      a contradiction with the hypothesis that $\frac{(K-1)}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$.

– **Case 2:** Suppose that

$$u_3^1 \leq u_1^1$$

so that we must have

$$P(\hat{q}_3) \geq P(\hat{q}_1).$$

If subjects in S_1^* report $\hat{q}_3$ with positive probability, then the expected payoff that a subject in S_1^* gets from reporting $\hat{q}_3$ must be greater than or equal the expected payoff he would get by (truthfully) reporting $\hat{q}_1$:

$$p^3_{\{3\}}u^1_{\{3\}} + p^3_{\{2,3\}}u^1_{\{2,3\}} + p^3_{\{3,4\}}u^1_{\{3,4\}} - 2c \geq p^1_{\{1\}}u^1_{\{1\}} + p^1_{\{1,2\}}u^1_{\{1,2\}} \tag{10}$$

But from part 1, we already know that we must have $P(\hat{q}_1) < P(\hat{q}_2)$, which, together with the condition $P(\hat{q}_3) \geq P(\hat{q}_1)$ implies that $u^1_{\{1\}} > u^1_{\{2,3\}}$. Therefore, inequality (10) implies that

$$\begin{aligned}
&p^3_{\{3\}}u^1_{\{1\}} + p^3_{\{2,3\}}u^1_{\{1\}} + p^3_{\{3,4\}}\overline{u} - 2c \geq p^1_{\{1\}}u^1_{\{1\}} + p^1_{\{1,2\}}\underline{u} \\
&\Rightarrow \left(1 - p^3_{\{3,4\}}\right)u^1_{\{1\}} + p^3_{\{3,4\}}\overline{u} \geq \left(1 - p^1_{\{1,2\}}\right)u^1_{\{1\}} + p^1_{\{1,2\}}\underline{u} + 2c \\
&\Rightarrow \left(1 - \frac{K-1}{\underline{S}}\right)u^1_{\{1\}} + \frac{K-1}{\underline{S}}\overline{u} \geq \left(1 - \frac{K-1}{\underline{S}}\right)u^1_{\{1\}} + \frac{K-1}{\underline{S}}\underline{u} + 2c \\
&\Rightarrow \frac{K-1}{\underline{S}} \geq \frac{2c}{\overline{u} - \underline{u}},
\end{aligned}$$

a contradiction with

$$\frac{K-1}{\underline{S}} < \frac{c}{\overline{u} - \underline{u}}.$$

3. $P(\hat{q}_i) < P(\hat{q}_{i+1})$ for all $i \in \{3, 4, \cdots, m-2\}$.
Let us prove by induction that $P(\hat{q}_i) < P(\hat{q}_{i+1})$ for all $i \in \{3, 4, \cdots, m-2\}$. Because we have already proven that

$$P(\hat{q}_1) < P(\hat{q}_2) < P(\hat{q}_3),$$

it suffices to show that, for any arbitrary $i \in \{3, 4, \cdots, m-2\}$,

$$P(\hat{q}_1) < P(\hat{q}_2) < \cdots < P(\hat{q}_i)$$

implies that

$$P(\hat{q}_i) < P(\hat{q}_{i+1}).$$

So for a given $i \in \{3, 4, \cdots, m-2\}$ suppose that $P(\hat{q}_1) < P(\hat{q}_2) < \cdots < P(\hat{q}_i)$ and suppose by way of contradiction that there is a NE in which $P(\hat{q}_i) \geq P(\hat{q}_{i+1})$. Then, either there is some $l > i+1$ with subjects in S_l^* reporting $\hat{q}_i$ with positive probability, or there is some $l < i$ with subjects in S_l^* reporting $\hat{q}_{i+1}$ with positive probability.

(a) Suppose that there is a $l > i+1$ such that (some) subjects in S_l^* report $\hat{q}_i$ with positive probability. Then, the expected payoff that subjects from S_l^* get from reporting $\hat{q}_i$ must be greater than or equal to their expected payoff from reporting $\hat{q}_{i+1}$:

$$\begin{aligned}
&p^i_{\{i\}}u^l_{\{i\}} + p^i_{\{i-1,i\}}u^l_{\{i-1,i\}} + p^i_{\{i,i+1\}}u^l_{\{i,i+1\}} - (l-i)c \geq \\
&p^{i+1}_{\{i+1\}}u^l_{\{i+1\}} + p^{i+1}_{\{i,i+1\}}u^l_{\{i,i+1\}} + p^{i+1}_{\{i+1,i+2\}}u^l_{\{i+1,i+2\}} - (l-i-1)c.
\end{aligned} \tag{11}$$

But because we are assuming $P(\hat{q}_1) < P(\hat{q}_2) < \cdots < P(\hat{q}_i)$, we must have

$$u^l_{\{i-1,i\}} > u^l_{\{i\}}. \tag{12}$$

Moreover, because we are assuming (by way of contradiction) that $P(\hat{q}_{i+1}) < P(\hat{q}_i)$, we must have

$$u^l_{\{i\}} < u^l_{\{i,i+1\}} < u^l_{\{i+1\}}. \tag{13}$$

Inequalities (11), (12) and (13) then imply that

$$\begin{aligned}
&\left(1 - p^i_{\{i-1,i\}}\right)u^l_{\{i,i+1\}} + p^i_{\{i-1,i\}}u^l_{\{i-1,i\}} \geq \left(1 - p^{i+1}_{\{i+1,i+2\}}\right)u^l_{\{i,i+1\}} + p^{i+1}_{\{i+1,i+2\}}u^l_{\{i+1,i+2\}} + c \\
&\Rightarrow \left(1 - \frac{K-1}{\underline{S}}\right)u^l_{\{i,i+1\}} + \frac{K-1}{\underline{S}}u^l_{\{i-1,i\}} \geq \left(1 - \frac{K-1}{\underline{S}}\right)u^l_{\{i,i+1\}} + \frac{K-1}{\underline{S}}u^l_{\{3,4\}} + c \\
&\Rightarrow \frac{K-1}{\underline{S}}(\overline{u} - \underline{u}) \geq c,
\end{aligned}$$

a contradiction with the hypothesis that $\frac{(K-1)}{\underline{S}} < \frac{c}{(\overline{u} - \underline{u})}$.

(b) Suppose that there is a $l < i$ such that (some) subjects in S_l^* report $\hat{q}_{i+1}$ with positive probability.

i. Suppose that
$$u^l_{\{i+1\}} > u^l_{\{l\}}.$$
Then, either one of the following conditions must hold:

A. There is a $j > i+1$ such that subjects from S_j^* report $\hat{q}_l$ with positive probability. This implies that
$$\begin{aligned} &p^l_{\{1\}}u^j_{\{l\}} + p^j_{\{l,l+1\}}u^j_{\{l,l+1\}} + p^j_{\{l-1,l\}}u^j_{\{l-1,l\}} - (j-l)c \geq \\ &p^{i+1}_{\{i+1\}}u^j_{\{i+1\}} + p^{i+1}_{\{i,i+1\}}u^j_{\{i,i+1\}} + p^{i+1}_{\{i+1,i+2\}}u^j_{\{i+1,i+2\}} - (j-i-1)c. \end{aligned} \tag{14}$$
But because we are assuming that $P(\hat{q}_1) < P(\hat{q}_2) < \cdots < P(\hat{q}_i)$, we must also have that
$$u^j_{\{l\}} > u^j_{\{l,l+1\}} > u^j_{\{i\}}.$$
Moreover, the assumption $P(\hat{q}_{i+1}) \leq P(\hat{q}_i)$ implies that
$$u^j_{\{i+1\}} \geq u^j_{\{i,i+1\}} \geq u^j_{\{i\}}.$$
Therefore, inequality (14) implies that
$$(1-p^l_{\{l-1,l\}})u^j_{\{l\}} + p^l_{\{l-1,l\}}u^j_{\{l-1,l\}} \geq (1-p^l_{\{i+1,i+2\}})u^j_{\{i\}} + p^{i+1}_{\{i+1,i+2\}}u^j_{\{i+1,i+2\}} + \underbrace{(i+1-l)}_{\leq 2}c$$
$$\Rightarrow \left(1-\frac{(K-1)}{\underline{S}}\right)u^j_{\{l\}} + \frac{(K-1)}{\underline{S}}\overline{u} \geq \left(1-\frac{(K-1)}{\underline{S}}\right)u^j_{\{i\}} + \frac{(K-1)}{\underline{S}}\underline{u} + 2c$$
$$\Rightarrow \underbrace{\left(1-\frac{(K-1)}{\underline{S}}\right)}_{>0}\underbrace{(u^j_{\{l\}} - u^j_{\{i\}})}_{<0} + \frac{(K-1)}{\underline{S}}(\overline{u}-\underline{u}) \geq 2c$$
$$\Rightarrow \frac{(K-1)}{\underline{S}}(\overline{u}-\underline{u}) \geq 2c$$
a contradiction with the hypothesis that $\frac{(K-1)}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$.

B. There is a $l < i-1$ such that (some) subjects from S_l^* report $\hat{q}_{i+1}$ with positive probability. This implies that
$$\begin{aligned} &p^{i+1}_{\{i+1\}}u^l_{\{i+1\}} + p^{i+1}_{\{i+1,i+2\}}u^l_{\{i+1,i+2\}} + p^{i+1}_{\{i,i+1\}}u^l_{\{i,i+1\}} - (i+1-l)c \geq \\ &p^l_{\{l\}}u^l_{\{l\}} + p^l_{\{l,l+1\}}u^l_{\{l,l+1\}} + p^l_{\{l-1,l\}}u^l_{\{l-1,l\}} \end{aligned} \tag{15}$$
But because we are assuming that $P(\hat{q}_1) < P(\hat{q}_2) < \cdots < P(\hat{q}_i)$, we must also have that
$$u^l_{\{l-1,l\}} > u^l_{\{l\}} > u^l_{\{l,l+1\}} \geq u^l_{\{i\}}.$$
Moreover, the assumption $P(\hat{q}_{i+1}) \leq P(\hat{q}_i)$ implies that
$$u^l_{\{i+1\}} \geq u^l_{\{i,i+1\}} \geq u^l_{\{i\}}.$$
Therefore, inequality (15) implies that
$$\left(1-p_{\{i+1,i+2\}^{i+1}}\right)u^l_{\{i+1\}} + p^{i+1}_{\{i+1,i+2\}}u^l_{\{i+1,i+2\}} \geq \left(1-p_{\{l,l+1\}^l}\right)u^l_{\{l\}} + p^l_{\{l,l+1\}}u^l_{\{l,l+1\}} + \underbrace{(i+1-l)}_{\geq 2}c$$
$$\Rightarrow \left(1-\frac{K-1}{\underline{S}}\right)u^l_{\{i+1\}} + \frac{K-1}{\underline{S}}\overline{u} \geq \left(1-\frac{K-1}{\underline{S}}\right)\underbrace{u^l_{\{l\}}}_{>u^l_{\{i\}}} + \frac{K-1}{\underline{S}}\underline{u} + 2c$$
$$\Rightarrow \left(1-\frac{K-1}{\underline{S}}\right)u^l_{\{i+1\}} + \frac{K-1}{\underline{S}}\overline{u} \geq \left(1-\frac{K-1}{\underline{S}}\right)u^l_{\{i\}} + \frac{K-1}{\underline{S}}\underline{u} + 2c$$
$$\Rightarrow \left(1-\frac{K-1}{\underline{S}}\right)\underbrace{\left(u^l_{\{i+1\}} - u^l_{\{i\}}\right)}_{\leq 0} + \frac{K-1}{\underline{S}}(\overline{u}-\underline{u}) \geq 2c$$
$$\Rightarrow \frac{K-1}{\underline{S}}(\overline{u}-\underline{u}) \geq 2c,$$
a contradiction with the hypothesis that $\frac{(K-1)}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$.

ii. So suppose that

$$u^l_{i+1} \leq u^l_l,$$

which implies that

$$P(\hat{q}_{i+1}) \geq P(\hat{q}_l).$$

As we are assuming subjects in S^*_l report $\hat{q}_{i+1}$ with positive probability, the expected payoff that a subject in S^*_l gets from reporting $\hat{q}_{i+1}$ must be greater than or equal the expected payoff he would get by (truthfully) reporting $\hat{q}_l$:

$$\begin{aligned}&p^{i+1}_{\{i+1\}}u^l_{\{i+1\}} + p^{i+1}_{\{i,i+1\}}u^l_{\{i,i+1\}} + p^{i+1}_{\{i+1,i+2\}}u^l_{\{l+1,l+2\}} - (i+1-l)c \geq \\ &p^l_{\{l\}}u^l_{\{l\}} + p^l_{\{l,l+1\}}u^l_{\{l,l+1\}} + p^l_{\{l-1,l\}}u^l_{\{l-1,l\}}\end{aligned} \tag{16}$$

But because we are assuming that $P(\hat{q}_1) < P(\hat{q}_2) < \cdots < P(\hat{q}_i)$, we must have that

$$u^l_{\{l-1,l\}} > u^l_{\{l\}}.$$

Moreover, the assumption $P(\hat{q}_{i+1}) \leq P(\hat{q}_i)$ together with the assumption $u^l_{i+1} \leq u^l_l$ imply that

$$u^l_{\{i,i+1\}} \leq u^l_{\{i+1\}} \leq u^l_{\{l\}}.$$

Therefore, inequality (16) implies that

$$\begin{aligned}&\left(1 - p^{i+1}_{\{i+1,i+2\}}\right)u^l_{\{l\}} + p^{i+1}_{\{i+1,i+2\}}u^l_{\{l+1,l+2\}} \geq \left(1 - p^l_{\{l,l+1\}}\right)u^l_{\{l\}} + p^l_{\{l,l+1\}}u^l_{\{l,l+1\}} + \underbrace{(i+1-l)}_{\geq 2}c \\ &\Rightarrow \left(1 - \frac{K-1}{\underline{S}}\right)u^l_{\{l\}} + \frac{K-1}{\underline{S}}\overline{u} \geq \left(1 - \frac{K-1}{\underline{S}}\right)u^l_{\{l\}} + \frac{K-1}{\underline{S}}\underline{u} + 2c \\ &\Rightarrow \frac{K-1}{\underline{S}}(\overline{u} - \underline{u}) \geq 2c,\end{aligned}$$

a contradiction with the hypothesis that $\frac{(K-1)}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$.

4. $P(\hat{q}_{m-1}) < P(\hat{q}_m)$.
Suppose by way of contradiction that there is a NE in which $P(\hat{q}_{m-1}) \geq P(\hat{q}_m)$. Then there must be some $l < m-1$, such that subjects in S^*_l report $\hat{q}_m$ with positive probability. For this to be true, the expected payoff from subjects in S^*_l of reporting $\hat{q}_m$ must be greater than or equal the expected payoff they would get from reporting $\hat{q}_{m-1}$. So we must have

$$\begin{aligned}&p^m_{\{m\}}u^l_{\{m\}} + p^m_{\{m-1,m\}}u^l_{\{m-1,m\}} - (m-l)c \geq \\ &p^{m-1}_{\{m-1\}}u^l_{m-1} + p^{m-1}_{\{m-2,m-1\}}u^l_{\{m-2,m-1\}} + p^{m-1}_{\{m-1,m\}}u^l_{\{m-1,m\}} - (m-1-l)c\end{aligned} \tag{17}$$

Because we are assuming that those who report $\hat{q}_{m-1}$ are more likely to be infected than those who report $\hat{q}_m$ (i.e., $P(\hat{q}_{m-1}) \geq P(\hat{q}_m)$), we must have

$$u^l_{\{m-1\}} < u^l_{\{m-1,m\}} < u^l_{\{m\}}.$$

Therefore, inequality (17) implies that

$$\begin{aligned}&u^l_{\{m\}} \geq (1 - p^{m-1}_{\{m-2,m-1\}})u^l_{\{m-1\}} + p^{m-1}_{\{m-2,m-1\}}u^l_{\{m-2,m-1\}} + c \\ &\Rightarrow p^{m-1}_{\{m-2,m-1\}}(u^l_{\{m-1\}} - u^l_{\{m-2\}}) \geq c \\ &\Rightarrow \frac{(K-1)}{\underline{S}}(\overline{u} - \underline{u}) \geq c \\ &\Rightarrow \frac{(K-1)}{\underline{S}} \geq \frac{c}{(\overline{u} - \underline{u})},\end{aligned}$$

a contradiction with the hypothesis that $\frac{(K-1)}{\underline{S}} < \frac{c}{(\overline{u}-\underline{u})}$.

$\blacksquare$

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