Search Steering in Two-Sided Platforms

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Abstract

Many two-sided platforms, such as Amazon and eBay, are known to steer customers towards certain products based on their willingness to pay. In this article, we study platforms' incentives to adopt this type of market segmentation and present conditions under which this can generate distortions that negatively impact the surplus from buyers and sellers. In our environment, a monopolistic platform matches sellers with buyers. Upon being matched, each pair of buyer and seller negotiates prices. If they choose to transact, the platform receives a commission fee proportional to the value of the transaction plus a flat fee per transaction. The platform is assumed to have full information about customers' and sellers' outside options. We show that, as long as the market is in excess supply or as long as *there is a crossing between the demand and supply curves*, the platform's optimal matching is suboptimal from the perspective of buyers and sellers.

Keywords: Market Segmentation, Information Design, Two-sided markets

1 Introduction

With the proliferation of online marketplaces, such as Amazon and eBay, many expected that information frictions in these markets would be mitigated, thus improving consumer welfare (e.g., Bakos (1997)). But theoretical research suggests that platforms may have incentives to obfuscate search (e.g., Diamond (1971) and Anderson and Renault (1999)) or provide biased search results (e.g., Bourreau and Gaudin (2022)) in a way that could harm customers' welfare. In this article, we analyze the latter type of incentive when commission fees are linear and increasing in prices, a common practice in many online marketplaces, such as Amazon.com, eBay.com and Upwork.com.[1] We show that, in an environment in which the platform can discriminate search results but cannot choose the prices of the goods sold by third-party sellers through its platform, the platform will tend to have incentives to prioritize finding feasible matches to more expensive products in order to increase the total number of transactions in the platform and inflate market prices (and thus the commissions it receives from sellers), in a process that hurts buyers and sellers. We also present some conditions under which the platform is more likely to implement these inefficient matching allocations.

This research is motivated by growing concerns that e-commerce websites may be discriminating search results based on users' data. As an example, a study in 2012 found that the travel website Orbitz offered

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[1]For Amazon's fee policy, see https://sellercentral.amazon.com/gp/help/external/200336920. For eBay, see https://pages.ebay.com/seller-center/seller-updates/2022-winter/fees-update.html. For Upwrok, see https://support.upwork.com/hc/en-us/articles/211062538-Freelancer-Service-Fees.

consistently higher prices for hotels to Mac users, as opposed to Windows users (Mattioli (2012)). Other studies, such as Hannak et al. (2014), have also found evidence that many e-commerce websites practice personalized search results as a function of customers' cookies and browsing history. More recently, empirical evidence of self-preferencing (the practice of granting more prominence to the platform's own products) has been found on Google Shopping (Amelio et al. (2018)) and Amazon.com (Farronato et al. (2023)). As yet another example, in Brazil the Online Travel Agency Decolar.com was fined in 2022 for charging significantly higher prices for the same hotels for customers based in Brazil compared to those based in Argentina, a practice known as *geopricing*.[2] There were also some instances in which the company made certain rooms that were available when searched from abroad, become unavailable when searched from Brazil.[3]

Driven by these findings, we build a theoretical framework in which a platform that intermediates transactions dictates one-to-one matches between buyers and sellers in exchange for a commission fee per sale, part of which is proportional to the final price chosen by the seller. Once matched, each buyer and seller negotiate prices through Nash Bargaining (similar to what happens in platforms such as Upwork.com or Redfin.com). We show that the platform has incentives to maximize the total number of transactions, which will tend to be greater than the number of transactions that would maximize buyers' and sellers' total surplus. The way the platform would create this excessive number of transactions is by allocating sellers with high production costs to customers with high willingness to pay, while allocating sellers with low production costs to customers with low willingness to pay, so as to increase market prices and, therefore, the commissions paid to the platform, and maximize the total number of transactions. So the platform essentially prioritizes finding feasible matches to more expensive products in order to 1) inflate the number of transactions and 2) inflate market prices in order to increase the commissions received per transaction. For such a policy to be feasible, the platform must have information on customers' willingness to pay, as well as sellers' cost structure or willingness to sell.

The intuition as to why a platform would want to implement such an inefficient matching algorithm can be best described with the help of a simple example. Consider an economy with 5 sellers and 5 buyers. Each seller is willing to sell only one unit of their product, and each buyer has unitary demand. Each seller $s_j \in \{s_1, s_2, \cdots, s_5\}$ sells a product with quality $q_j \geq 0$, with a production cost of $c_j \geq 0$, where c_j could also be interpreted as the seller's outside option, i.e., the price at which they can sell their product outside the platform. Each buyer $b_i \in \{b_1, b_2, \cdots, b_5\}$ has an outside option $u_i \geq 0$. In figure 1 we order sellers' net value, $q_j - c_j$, in descending order, and buyers' outside option, u_i, in ascending order.

From figure 2, we can see that the total surplus from this economy is maximized when buyers b_1, b_2, b_3 (i.e., those with highest willingness to pay) transact with sellers s_1, s_2, s_3 (those who generate most net value), and the remaining buyers and sellers do not transact, so that only 3 transactions take place in the economy. In this case, the total surplus could be approximated by the blue region from the graph.

But the platform could, in principle, induce 5 transactions by matching each buyer $b_i \in \{b_1, b_2, \cdots, b_5\}$ with seller s_{6-i}. This matching is displayed in figure 2 by the curved arrows. Because in each match the net value created by the seller surpasses the customer's outside option, each match should yield a transaction, thus generating a total surplus given by the blue area minus the deadweight loss given by the grey area in the figure. So, if the platform only collected a fixed fee per transaction that did not depend on prices, this matching allocation would generate a greater revenue to the platform than the Pareto efficient matching.

[2] Source: https://gcalaw.com.br/en/geopricing-and-geoblocking-are-considered-a-violation-of-consumer-rights-by-senacon/

[3] Source: https://brnarede.com.br/en/brazils-decolar-com-is-in-hot-water-for-geopricing-allegations/ and https://oglobo.globo.com/boa-viagem/entenda-que-geopricing-como-hoteis-no-exterior-podem-estar-cobrando-mais-caro-de-brasileiros-1-25077743.

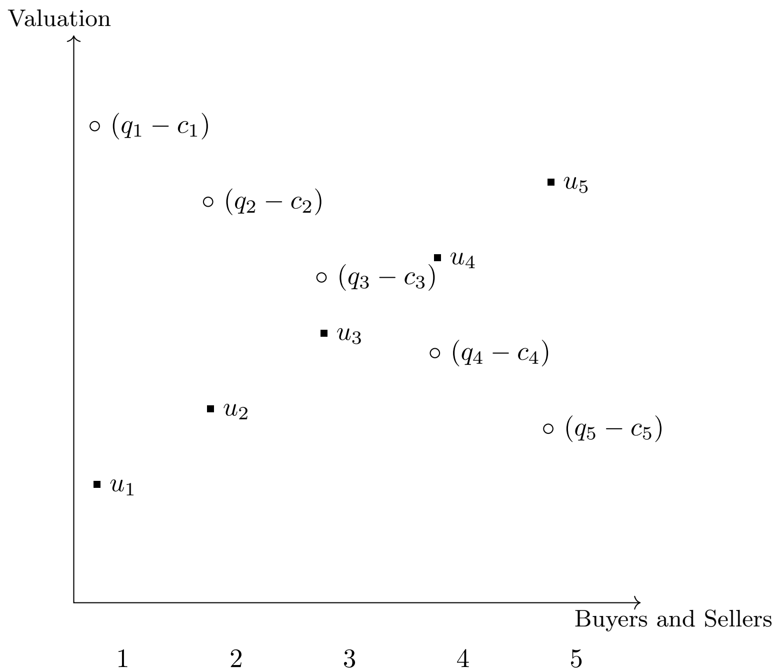


Figure 1: A plot depicting the net value by each seller, and the outside option from each buyer.

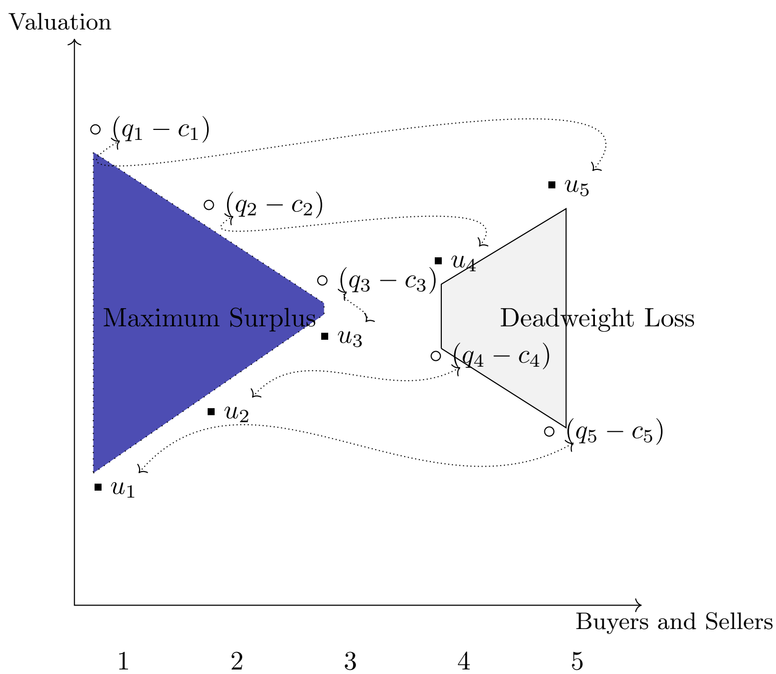


Figure 2: The plot depicts an approximation of the total surplus obtained when agents are matched so as to maximize the total number of transactions.

The platform would have even more incentives to implement this matching if commissions were linear and increasing in prices, as doing so would inflate market prices, thus resulting in higher commissions per sale. Indeed, in this example, buyer b_1 would end up paying more if he were to negotiate with seller s_5 instead of with seller s_1, as seller s_5 has a higher cost. Therefore, it is in the platform's best interest to secure a feasible match to seller s_5, as doing so would result in a higher price and, therefore, a higher commission than the one it can get from seller s_1.

We also derive conditions under which the platform's optimal policy does not maximize the sum of buyers' and sellers' surplus. Indeed, we show that if there are more sellers than buyers,[4] or if the demand curve crosses the supply curve, as in figure 1, the platform's optimal matching will not maximize buyers' and sellers' total surplus. Intuitively, in these two scenarios the platform cannot possibly get a transaction from every seller, in which case it has to prioritize forming good matches to sellers with high production costs so as to inflate market prices, thus increasing the commissions received by the platform from each transaction. Because these conditions are rather mild, many markets are expected to be affected by this inefficiency.

Ideally, the platform would want to implement a matching that is stable, i.e., a matching in which there is no pair of buyer and seller who could form a blocking coalition, as stability prevents customers from clinching matches that were intended to someone else, in a process that could drastically reduce the overall number of transactions in the platform. Stability becomes less of a concern to the platform when customers have a high search cost, as in this case customers are more inclined to follow through with the platform's recommendation to avoid a costly search. We show that matchings that maximize the surplus of buyers and sellers are always stable, whereas alternative profit-maximizing matchings are not necessarily stable when the search cost is sufficiently low. This result indicates that one factor that may nudge the platform to adopt an efficient matching are regulatory policies that prevent the platform from obfuscating search (e.g., through drip pricing).

Notice that in our environment price negotiations between buyers and sellers happen after the match is determined by the platform. Though this is more consistent with market structures in which buyers and sellers are allowed to bargain after they are matched with one another (such as Upwork.com, Realtor.com or Redfin.com), our results also apply to situations in which sellers first commit to a price, and then the platform determines the matching, and sellers are not sophisticated enough to predict the effect of their prices on the match that they get from the platform; or to situations in which, though sellers can predict the effect of prices on the matching assignment, they are bound to charge a price (net of commission fees) less than or equal to the price they practice outside of the platform lest they suffer public criticism for practicing price discrimination, or violate some price parity clause imposed by the platform. In the latter case, seller s_i would charge a price equal to his outside option c_i, i.e., equal to the price it can sell its product outside of the platform, plus the commissions charged by the platform. Additionally, though the results derived in this article have been framed in terms of a two-sided platform, it also applies to other forms of intermediation, such as the ones conducted by real estate agencies.

Also, notice that the source of inefficiency in our model comes from the fact that the platform does not internalize the costs of manufacturing more expensive products. Indeed, if the platform was the owner and manufacturer of the products sold, prioritizing the sale of more expensive products would not be ideal, as the platform would have to bear the cost of manufacturing these more expensive products.

The reminder of the paper is structured as follows: In section 2 we provide a summary of the related literature. In section 3 we describe the environment from our model. In section 4 we derive the platform's optimal matching policy and the conditions under which the final matching allocation chosen by the platform will not maximize buyers' and sellers' welfare. Section 5 characterizes the platform's optimal matching

[4] Here we are only considering sellers and buyers who have at least one feasible match, i.e., agents for which there is at least one other agent, on the other side of the market, with whom they would be willing to transact.

allocation in terms of stability and discusses policy implications of this result. Section 6 concludes.

2 Related Literature

The theoretical literature on incentives of e-commerce platforms is filled with examples that show how platforms may be willing to create information frictions that can potentially hurt customers, such as the seminal work of Diamond (1971) and Anderson and Renault (1999). In their theoretical model, customers sequentially search for a product through *undirected search*. This means that, in each period, customers are randomly matched with a product and decide whether to buy it at the advertised price or keep searching for a better match. Like in our environment, they assume that prices of the final good are set by the third-party sellers, so that the platform cannot directly control the final prices of the products in the market. They show that, if there are infinitely many sellers operating in the platform selling the same homogeneous product, then, for any given search cost, no matter how small, there is an equilibrium in which all sellers choose the monopoly price. This result, known as the Diamond Paradox, is arguably very puzzling, as one would expect that competition in a market with many providers selling the same homogeneous product would drive prices down to the perfectly competitive level.

An implication of this result is that, if part of the platform's commissions are proportional to the price of the product sold (as it is common in many two-sided platforms, such as Amazon, eBay and Airbnb), then the platform may have incentives to foster search frictions to increase the final price charged by sellers. This observation has led others to present specific mechanisms through which platforms can increase search costs, such as the work of Eliaz and Spiegler (2011) and Casner (2020), who show that a platform may have incentives to allow some low quality sellers to enter the market in order to obfuscate search, or the work of Ellison (2005) and Ellison and Ellison (2009) who show that in a competitive environment firms may be able to achieve higher profits when add-on prices (e.g., shipping fees, resort fees, etc.) are hidden. These results illustrate how two-sided platforms may sometimes be willing to add inefficiencies into their market to regain some control over the prices chosen by third-party sellers. These studies assume, however, that the platform is not in control over who gets matched with whom in the search process. In contrast, in our environment, the platform is the one determining the matches, and it uses this ability as a tool to regain some control over market prices.

In a different approach, Hagiu and Jullien (2011) study a search model in which, instead of the search results being random, the platform is the one dictating the order of the search results from each customer. In their specification, there are two sellers who intermediate transactions through the platform, each selling a different product. Customers are willing to buy more than one of the products, and sellers face no capacity constraints.[5] They also assume that the platform has no incentives to increase market prices as a means to increase revenues, as their source of income are the number of visits to each seller's webpage or number of sales. So, contrary to Diamond (1971) and Anderson and Renault (1999), in their specification high search

[5]Though Hagiu and Jullien (2011) provide an extension in their companion paper that allows products to exhibit some degree of substitutability, it should be noticed that, in their environment, when products are perfect substitutes and the platform only charges a flat fee τ per transaction, the platform has no incentives to steer customers towards the "*wrong*" product. This is not necessarily true in our environment where sellers face capacity constraints *or* the platform earns revenues proportional to the products' final price. For example, in digital markets (e.g., App Store or Google Play) most sellers face virtually no capacity constraint, but the platform may have incentives to systematically recommend expensive products if the commissions per transaction are proportional to the product's final price. Similarly, in a housing market where the intermediary only charges a flat fee per transaction, the intermediary may find it preferable to steer a customer with a high willingness to pay to a more expensive house in order to save the cheap ones for those with low willingness to pay, thus increasing the overall number of transactions.

costs unambiguously hurt the platform, as they cause customers to inspect fewer products, reducing their overall consumption in the extensive margin. In our environment, on the other hand, products are perfect substitutes, so each customer is only willing to purchase one product, and sellers face capacity constraints (a feature that is common in the housing and job markets). Moreover, the platform earns a commission that is proportional to the price of the product sold, in which case the platform may have incentives to limit customers' search ability to avoid competition between sellers, as competition causes equilibrium prices to go down, which reduces the commissions received by the platform. Therefore, in our environment what motivates the platform to steer customers towards certain products is its desire to increase purchases on the extensive margin (but not on the intensive margin) due to sellers' limited production capacity, and also to inflate equilibrium prices. In addition, we provide a fuller characterization of the welfare implications that search steering may have on buyers and sellers.

This article is also related to the literature on double auctions (i.e., auctions that intermediate the transactions between buyers and sellers), such as the work of Deshmukh et al. (2002), Ausubel et al. (2017) and Satterthwaite et al. (2022). But because the auctioneer is usually interested in implementing efficient and/or incentive compatible mechanisms, this literature has usually proposed and studied auction formats in which the sellers with lowest opportunity costs are the ones who end up transacting (assuming that the products sold are homogeneous) as it is the case in the uniform and pay-as-bid auctions. Indeed, both the uniform and pay-as-bid auctions are Pareto efficient when participants submit their true valuations, and the auctioneer does not collect distortionary commission fees. Though neither mechanism is strategy-proof (e.g., see Ausubel et al. (2014) or Binmore and Swierzbinski (2000)), the uniform price auction is known to be "*almost*" strategy-proof when the number of participants in the market is large, and each participant has a small market share on their side of the market (Azevedo and Budish (2019)).

More in line with our approach, Zhao et al. (2011) assumes that the auctioneer already knows the valuations from buyers and sellers, so that it can abstract from incentive compatibility issues. They propose the most efficient algorithm to match buyers with sellers among the ones that maximize the total number of transactions in the economy. This algorithm is attractive if the auctioneer collects a fixed revenue per transaction that does not depend on the final price at which the product is sold. This algorithm can generate more transactions than the socially optimum. In our specification, on the other hand, we assume that the platform (i.e., the auctioneer) also collects a fee per transaction that is proportional to the final price at which the product is sold. So, we predict that the matching mechanism used by the platform will be even more inefficient than the one proposed by Zhao et al. (2011), as the platform will not only try to maximize the total number of transactions, but it will also prioritize finding eligible buyers to sellers with a high outside option (i.e., with high cost), so as to inflate the final prices paid by customers, and therefore, the fees received by the platform.

Similarly, Yasuda (2016) shows that, in a decentralized market for a homogeneous good, the competitive equilibrium allocation in which every seller charges the market clearing price minimizes the total number of transactions in the economy. Therefore, as in Zhao et al. (2011), Yasuda (2016) concludes that, if this market had a centralized intermediator (not necessarily an auctioneer, but a matching platform), the intermediator would probably have incentives to deviate from the competitive equilibrium to increase the volume of transactions and, therefore, the fees received per transaction. We extend this result by allowing the platform to extract rent proportional to the price from each transaction. We also fully characterize the optimal policy adopted by the platform.

The research most closely related to our work is probably Boerner and Quint (2023). The authors build an environment similar to Zhao et al. (2011) and Yasuda (2016) but allow the intermediary to charge a fee that is linear in prices, which is more consistent with the pricing policy adopted by many digital platforms. However, while they focus most of their analysis on providing conditions under which the platform's optimal policy is more efficient than matching agents randomly, most of our results point towards the opposite direction: that the platform's optimal policy tends to harm buyers and sellers if we use as our benchmark a matching that maximizes buyers' and sellers' surplus (instead of random matching). We examine an alternative benchmark due to our conviction that random matching sets an exceedingly modest standard for what a platform ought to achieve to maximize the welfare of buyers and sellers. This is particularly relevant given the substantial market power wielded by numerous two-sided platforms. Moreover, the sufficient conditions derived in Boerner and Quint (2023) that guarantee that the platform's optimal policy performs well in terms of maximizing total surplus only hold asymptotically when the market is large and in expectation, and they require the market to be unbalanced. Meanwhile, the conditions we derive that guarantee that the platform's optimal matching allocation *does not* maximize buyers' and sellers' surplus are substantially milder and do not require market thickness: we only require the market to have more sellers than buyers (i.e., the market to be unbalanced) *or* that the "demand crosses supply" (see section 1 for details). Moreover, we show how to generalize the simulations conducted in Boerner and Quint (2023) to allow for horizontal differentiation by making use of the Hungarian method, and we analyze the effect of search costs on the platform's incentives to deviate from implementing a stable (and therefore, more efficient) matching.

Teh and Wright (2022) also develop an environment similar to ours, but in which the third-party sellers are the ones who set the commissions per transaction paid to the platform. Sellers who elicit a higher commission are more likely to be recommended to customers in a process that resembles ad auctions conducted by search engines, such as Google. They find that if sellers are able to elicit the commissions they pay to the platform, they will end up charging higher equilibrium prices, as sellers pass on some of the commission costs to customers, a phenomenon that is also present in our model. This increase in prices reduces the overall number of transactions in equilibrium (as an increase in prices reduces demand), thus creating a deadweight loss. Despite this inefficiency, Teh and Wright (2022) find that, in equilibrium, the platform implements an unbiased matching, i.e., it makes the same recommendations as the ones it would make if its objective was to maximize the total expected surplus from each match. But in our environment in which the commissions paid to the platform are linear in prices (a feature that is present in many two-sided platforms), this may not be the case, as the platform would have incentives to prioritize finding suitable matches to more expensive products, which are not necessarily the products that, if sold, would generate more surplus.

In Bourreau and Gaudin (2022), content providers make royalty bids to a streaming platform (e.g., Netflix) that charges a flat fee for customers. Naturally, the platform has incentives to give more prominence to sellers who require less royalties per watch (i.e., who make more competitive bids). In equilibrium, content providers with lower marginal costs make more competitive bids and, therefore, receive more prominence, which, like in Teh and Wright (2022), results in a more efficient allocation. In our environment, on the other hand, where the commission fee paid to the platform is linear in prices, and consumers must pay for each additional product consumed, we find that the platform has incentives to promote more expensive products, which is inefficient.

Zhou and Zou (2023) developed an environment with two sellers and a platform in charge of recommending them to a continuum set of buyers. They assume products to be horizontally differentiated. The platform observes each buyer's signal regarding his preference and, based on this signal and based on sellers' prices, recommends a product to the buyer. Therefore, different from our environment and the ones from Hagiu and

Jullien (2011) and Boerner and Quint (2023), they allow sellers' pricing decisions to influence the platform's recommendations. Like in Hagiu and Jullien (2011), sellers face no capacity constraints, so the platform does not need to ration "*easy sales*". However, the platform should ensure that the equilibrium prices set by sellers are not too low, as their commissions per sale are proportional to the transacted price. They show that, in some instances, the platform could be made better off if it was able to commit to not use prices as a recommendation criterion, as this would prevent sellers from undercutting each other's prices to appear at the top of search results. They only derive results, however, for a case in which the market is comprised of only two firms, whereas our results apply to situations in which the market has multiple firms with capacity constraints. Moreover, while they rely on the assumption that the market has a continuum set of customers, we follow the mechanism design approach by assuming that there is a discrete set of customers and sellers, which not only is more realistic, but allows variants of the algorithms that we present to be more easily implemented in practical applications.

This article is also related to the literature on market segmentation, such as the work of Bergemann et al. (2015), Haghpanah and Siegel (2019) and Yang (2022). This literature, however, usually assumes that the seller segmenting the market is in control over the prices of each one of its products and that it internalizes production costs. In such an environment, market segmentation usually has the effect of improving total welfare. As an example, for the extreme case in which the seller practices perfect price discrimination by charging each customer their willingness to pay, total surplus is maximized, though in this case customers earn zero surplus. In our environment, on the other hand, the platform intermediating transactions between buyers and sellers cannot dictate the prices charged by sellers. As a result, attempts by the platform to segment the market in its favor will generate inefficiencies not predicted by the previous literature.

3 Model

A market is comprised of a set of n buyers, $B \equiv \{b_1, b_2, \cdots, b_n\}$, and a set of m sellers, $S \equiv \{s_1, s_2, \cdots, s_m\}$. Buyers have a unitary demand for the products sold by the sellers, and each seller can only sell one unit of their product to a single customer. For each $i \leq n$ and each $j \leq m$ we denote $q_{ij} \in \mathbb{R}_+$ as how much buyer $b_i \in B$ values the product from seller $s_j \in S$. We denote $c_j \in \mathbb{R}_+$ as the cost or the outside option from seller s_j of selling its product. We denote u_i as the outside option from buyer i (i.e., the utility that buyer i gets if he does not purchase any product from the platform).

Without loss of generality, we order buyers and sellers in ascending order of their outside options and costs, respectively. That is, unless specified otherwise, we assume that

$$u_1 \leq u_2 \leq \cdots \leq u_n \tag{1}$$

and

$$c_1 \leq c_2 \leq \cdots \leq c_m. \tag{2}$$

A monopolistic platform is responsible for intermediating transactions between buyers and sellers. More precisely, the platform will match customers to sellers through a matching function μ. In this environment we will only allow for one-to-one matches, i.e., each seller can be matched with at most one customer and vice versa. The matching function μ will map agents on one side of the market to agents on the other side of the market, with the possible exception of an agent being matched with himself to account for the possibility that the agent remains unmatched.

Definition 1. *A matching μ is a function mapping $B \cup S$ into itself, such that:*

i) $\forall b \in B$, $\mu(b) \in S \cup \{b\}$.

ii) $\forall s \in S$, $\mu(s) \in B \cup \{s\}$.

iii) $\forall b \in B$ *and* $s \in S$, $s \in \mu(b)$ *if and only if* $\mu(s) = b$.

To simplify the exposition, we sometimes also use the notation $s_j : b_i$ to indicate that seller s_j is matched to buyer b_i. In other words, $s_j : b_i$ is equivalent to writing $\mu(s_j) = b_i$ or $\mu(b_i) = s_j$.

Once the matching is determined, matched agents can transact. Suppose that the platform receives a fixed share $\psi \in [0, 1]$ of the price paid by the customer for each realized purchase, plus a flat commission fee $\tau \geq 0$ per transaction. Prices are determined ex-post, after the matches are formed, and after agents learn about the commission fees ψ and τ charged by the platform. In practice, prices could be determined after the matches take place if buyers and sellers are allowed to bargain after they are matched with one another, as in Upwork.com or Redfin.com.

We assume that prices are determined through Nash Bargaining, where each buyer has bargaining power $\alpha \in [0, 1]$ and each seller has bargaining power $1 - \alpha$. More precisely, if $\psi < 1$ and seller s_j is matched with a buyer b_i such that $(1 - \psi)(q_{ij} - u_i) \geq c_j + \tau$, then a transaction will occur, and the seller will set its price at

$$\begin{aligned} p_j^*(b_i, \psi, \tau) &= \underset{p_j \in \left[\frac{c_j+\tau}{1-\psi}, q_{ij}-u_i\right]}{\arg\max} (q_{ij} - p_j - u_i)^{\alpha}((1 - \psi)p_j - c_j - \tau)^{1-\alpha} \\ &= (1 - \alpha)(q_{ij} - u_i) + \alpha \frac{c_j + \tau}{1 - \psi}, \end{aligned} \tag{3}$$

generating a revenue of

$$\psi p_j^*(\psi, \tau, b_i) + \tau = \psi \left[(1 - \alpha)(q_{ij} - u_i) + \alpha \frac{c_j + \tau}{1 - \psi}\right] + \tau \tag{4}$$

to the platform. If $(1 - \psi)(q_{ij} - u_i) < c_j + \tau$, then no transaction transpires from the match between seller s_j and buyer b_i, so that the platform earns zero revenues from this match. If $\psi = 1$ and $c_j = \tau = 0$, the seller is indifferent between not selling and selling its product at any price. To guarantee that the profit function is upper semicontinuous, we will assume that in this case the seller will charge $p_j = \max\{q_{ij} - u_i, 0\}$.[6]

Notice that, when $\alpha = 1$, i.e., when buyers have all bargaining power, seller s_j elicits the price $(c_j + \tau)/(1 - \psi)$ regardless of his match. Because in this case the matching algorithm chosen by the platform does not affect sellers' pricing strategy, we could invert the timing of the model by assuming that sellers first elicit a price, and then, based on those prices, the platform decides who gets matched with whom, which would be more consistent with marketplaces such as Amazon and eBay. In practice, this could happen if, for instance, the platform imposes a price parity clause that prohibits each seller s_j from charging a price, net of commission fees, higher than the price c_j at which he sells its products outside of the platform (i.e., it prohibits each seller s_j from charging a price above $(c_j + \tau)/(1 - \psi)$).[7]

[6]The hypothesis of upper semicontinuity of the objective function with respect to (ψ, τ) guarantees the existence of an optimal (ψ, τ) for any given match μ (see proposition 6).

[7]Notice that c_j is the **net payoff** that the seller would get if he decided to sell his product outside of the platform, not necessarily the price it would charge. Assuming that selling the product outside the platform also entails paying similar commissions, the price outside the platform would be $(c_j + \tau)/(1 - \psi)$ and not c_j.

Taking the commission rates (ψ, τ) as given, we define, for each matching function μ,

$$FM_\mu \equiv \{(s_j, b_i) \in S \times B; \mu(s_j) = b_i \text{ and } (1-\psi)(q_{ij} - u_i) \geq c_j + \tau\},$$

i.e., FM_μ represents all the matches associated with μ that generate transactions.

Once a matching μ is formed it generates a total surplus to buyers and sellers equal to

$$TS(\mu) \equiv \sum_{(s_j, b_i) \in FM_\mu} (q_{ij} - \psi p_j^*(\psi, \tau, b_i) - c_j - u_i - \tau).$$

For a given pair of commissions (ψ, τ), we say a matching is constrained Pareto efficient (CPE) if it maximizes $TS(\mu)$. As agents in our economy have quasilinear utility, this definition of efficiency is equivalent to the notion that no seller or buyer can be made better off without making another buyer or seller worse off. Notice that this definition does not include the surplus extracted by the platform, as our main interest is in evaluating the platform's optimal policy on consumers' and sellers' total surplus. Also notice that this definition takes the fees as exogenous, so it also does not take into account the distortionary effects caused by the fees (ψ, τ). The reason we use this second-best notion of Pareto efficiency is because, if $\psi > 0$ or $\tau > 0$, the final allocation would almost always be inefficient for any matching μ due to the distortionary effects caused by the fees. So, the only way to guarantee efficiency would be to completely eliminate the fees charged by the platform, which is not a very practical solution. That being said, our results regarding which matchings are CPE also apply to the case in which $\psi = \tau = 0$, i.e., the case in which our definition of CPE is equivalent to the (first-best) notion of Pareto efficiency.

Definition 2. *(Constrained Pareto Efficiency) For a given (ψ, τ), we say a matching μ is Constrained Pareto Efficient (CPE) if*

$$TS(\mu) \geq TS(\mu')$$

for all possible alternative matchings μ'.

In the next section we characterize the platforms' optimal matching μ in terms of efficiency and number of transactions, taking its commission fees (ψ, τ) as given.

4 Optimal Matching with exogenous commissions

In this section we will consider the problem of finding the optimal matching taking the commission fees, ψ and τ, as given. This hypothesis is consistent with current market practices, as commission rates are usually not adjusted very frequently over time (Zhou and Zou (2023)).

So we will characterize the solution to the following problem:

$$\max_\mu \tau |S_\mu| + \psi \sum_{s_j \in S_\mu} p_j^*(\mu(s_j), \psi, \tau), \tag{5}$$

where

$$S_\mu \equiv \{s_j \in S; \mu(s_j) = b_i \in B \wedge (1-\psi)(q_{ij} - u_i) \geq c_j + \tau\}$$

corresponds to the set of sellers who end up transacting given the fees ψ and τ and matching function μ, and $p_j^*(\mu(s_j), \psi, \tau)$ is the equilibrium price given by equation (3).

We first analyze the problem when all products have the same quality, i.e., when $q_{ij} = \overline{q}$ for all $i \le n$ and all $j \le m$. Then we discuss how to find the optimal matching when products are differentiated. The Appendix provides an extension showing how to find the optimal commissions together with the optimal matching allocation.

4.1 Homogeneous products

In this section we will characterize Constrained Pareto Efficient (CPE) matchings and profit maximizing matchings when all products have the same quality. We will show that, in this case, finding a CPE matching is trivial: one only needs to ensure that the sellers and buyers transacting are the ones with lowest outside options. A simple algorithm can be used to find the platform's optimal matching for a given $\psi \in [0,1]$ and $\tau \ge 0$. We show that this algorithm not only maximizes the profits of the platform, but also the number of transactions that can be made, given the commissions (ψ, τ). We also present necessary and sufficient conditions under which the matching obtained through this algorithm is *not* CPE.

Suppose that all products have the same quality, i.e., $q_{ij} = \overline{q} \in \mathbb{R}_+$ for all $i \le n$ and all $j \le m$. Then, from equation (4) we have that the platform's revenue for a match between a seller s_j and a customer b_i such that $(1-\psi)(\overline{q} - u_i) \ge c_j + \tau$, is given by

$$\psi \left[(1-\alpha)(\overline{q} - u_i) + \alpha \frac{c_j + \tau}{1-\psi} \right] + \tau. \tag{6}$$

In this case, Constrained Pareto Efficiency is achieved by ensuring that the customers with the lowest outside options and the sellers with the lowest costs transact. Indeed, it is a well-known result that in an economy with transfers where agents have quasi-linear utility, efficiency is maximized when agents (buyers and sellers) located on the left-hand side of the point where the supply curve crosses the demand transact, generating the total surplus in blue depicted in figure 2. After the introduction of distortionary taxes (in our case, commissions), the same principle applies to the shifted supply and demand to give us a second-best allocation. Clearly, one way to achieve this second-best allocation is by implementing Positive Assortative Matching (PAM), i.e., by matching buyers with a high willingness to pay with sellers with a high willingness to sell. But a permutation of the matching between these same buyers and sellers should generate the same amount of total surplus with some redistribution of wealth among them. These results are formally stated in proposition 1 and corollaries 1 and 2 below.

Proposition 1. *(Necessary and sufficient condition for μ to be CPE) Suppose that $q_{ij} = \overline{q} \in \mathbb{R}_+$ for all $i \le n$ and all $j \le m$. A matching μ is Constrained Pareto Efficient (CPE) if and only if each seller s_j such that $j \le \min\{n, m\}$ and $(1-\psi)(\overline{q} - u_j) \ge c_j + \tau$ is matched to a buyer b_i such that $(1-\psi)(\overline{q} - u_i) \ge c_i + \tau$.*

Proof: Proof in the Appendix. ∎

Corollary 1. *(PAM is CPE) Suppose that $q_{ij} = \overline{q} \in \mathbb{R}_+$ for all $i \le n$ and all $j \le m$. If $\mu(s_i) = b_i$ for all $i \le \min\{n, m\}$, then μ is CPE.*

It follows directly from proposition 1 that a *CPE* matching μ maximizes the number of transactions subject to the constraint that every seller who transacts under μ must also be willing to transact with any other buyer who transacts under μ.

Corollary 2. *(CPE matchings maximize the number of transactions subject to a constraint) Suppose that* $q_{ij} = \overline{q} \in \mathbb{R}_+$ *for all* $i \leq n$ *and all* $j \leq m$*. If a matching* μ *is CPE, then*

$$\mu \in \arg\max_{\mu'} |FM_{\mu'}|$$
$$s.t. \quad (1-\psi)(\overline{q} - u_{i'}) \geq c_j + \tau \quad and \quad (1-\psi)(\overline{q} - u_i) \geq c_{j'} + \tau \quad \forall (s_j, b_i), (s_{j'}, b_{i'}) \in FM_{\mu'}$$

Proof: Proof in the Appendix. ■

In this environment one can easily find instances in which the platform has no incentives to implement a *CPE* matching, as illustrated in the following example.

Example 1. *(CPE matchings do not always maximize profits) For a given commission fee,* $\psi \in [0, 1]$ *and* $\tau \geq 0$*, the optimal match* μ^* *chosen by the platform is not necessarily CPE.*

Indeed, consider a market where the set of sellers is given by $\{s_1, s_2, s_3, s_4\}$*, and the set of buyers is given by* $\{b_1, b_2, b_3, b_4\}$*. Suppose that* $q_{ij} = 20\ \forall i, j \leq 4$*, and that sellers' costs are given by*

$$(c_1, c_2, c_3, c_4) = (2, 8, 14, 16),$$

while customers' outside options are given by

$$(u_1, u_2, u_3, u_4) = (0, 3, 5, 7).$$

In a Pareto Efficient allocation, buyers b_1*,* b_2 *and* b_3 *should be each matched with one of the first three sellers with highest net quality (i.e., with sellers* s_1*,* s_2 *and* s_3*). But if the platform chooses* $(\psi, \tau) = (1/2, 0)$*, then all of the Pareto efficient matchings would generate a total profit of* 9.75 *for the platform, whereas if the platform implemented the matching*

$$s_1 : b_4, \quad s_2 : b_3, \quad s_3 : b_2, \quad s_4 : b_1$$

its profits would equal 12.25.

Intuitively, the reason why the platform may not want to implement a *CPE* matching can be explained as follows. Rearranging expression (6), we can write the platform's revenue for a match between a seller s_j and a customer b_i such that $(1-\psi)(\overline{q} - u_i) \geq c_j + \tau$, as

$$\psi(1-\alpha)(\overline{q} - c_j - \tau - u_i) + \frac{\psi(1 - \psi + \alpha\psi)}{1-\psi}(c_j + \tau) + \tau. \tag{7}$$

Though the first term from expression (7) is increasing in the total surplus created from the transaction between seller s_j and buyer b_i, $(\overline{q} - c_j - u_i)$, the sum of the last two terms from this expression is increasing in the opportunity cost from selling, c_j, and on τ. So the platform is not exclusively interested in maximizing the creation of value, but also in maximizing the total number of transactions so as to collect more revenues through τ, and also on facilitating the transaction of more expensive products, as that increases the final price charged by sellers, and therefore the commissions collected through ψ.

So now we provide an algorithm for finding the profit-maximizing matching μ, taking the commission fees (ψ, τ) as given. This algorithm, independently obtained by Boerner and Quint (2023), can be described as follows: we start by matching the "worst seller" (the one with highest cost) with the "best customer" (the one with lowest outside option) and keep doing this iteratively with the agents who have not been matched yet.

Algorithm 1. *For each $B' \subseteq B$, each $s_j \in S$ and each $\psi \in [0,1]$ and $\tau \geq 0$ define*

$$F(s_j, B', \psi, \tau) \equiv \{b_i \in B'; (1-\psi)(q_{ij} - u_i) \geq c_j + \tau\},$$

i.e., $F(s_j, B', \psi, \tau)$ is the set of "feasible buyers" in B' for seller s_j, or more precisely, it is the set of buyers in B' who, if matched with seller s_j, would end up purchasing their product.

i) Initialize $j = m$ and define $B_j = B$.

ii) Compute $F(s_j, B_j, \psi, \tau)$. If $F(s_j, B_j, \psi, \tau) \neq \emptyset$, set $\mu(s_j)$ equal to $b_i \in F(s_j, B_j, \psi, \tau)$ such that $u_i \leq u_l$ for all $u_l \in F(s_j, B_j, \psi, \tau)$ (i.e., match this seller with the buyer with lowest outside option who has not been matched yet) and define $B_{j-1} = B_j \backslash \{b_i\}$, else set $\mu(s_j) = s_j$ (i.e., keep seller s_j unmatched) and define $B_{j-1} = B_j$. Then proceed to the next step.

iii) If $j > 1$, redefine $j = j - 1$ and repeat step ii), else, stop the algorithm.

Theorem 1. *(Algorithm 1 maximizes profits and number of transactions) Suppose that $q_{ij} = \overline{q} \in \mathbb{R}_+$ for all $i \leq n$ and all $j \leq m$. Then, given $\psi \in [0,1]$ and $\tau \geq 0$, the matching allocation obtained from algorithm 1:*

i) Maximizes the number of transactions that can be generated in the economy, i.e., it maximizes $|S_\mu|$.

ii) Maximizes the profits of the platform, i.e., it solves the maximization problem (5).

Proof: In the Appendix. ∎

To some extent, algorithm 1 does the opposite of what *PAM* does. Indeed, while *PAM* attempts to secure good matches to low-cost sellers, algorithm 1 attempts to secure good matches to high-costs sellers. The platform has incentives to adopt such an algorithm for two reasons: 1) high-cost sellers charge higher prices, so they generate more revenues to the platform in the event they transact (assuming $\psi > 0$); 2) moreover, this procedure generates more transactions (and therefore more revenues), as it frees low-cost sellers to transact with buyers who would otherwise not transact if matched with a high-cost seller.

So, we have seen that, if the products sold are homogeneous, algorithm 1 will always yield an optimal matching to the platform, which is not always *CPE* (see example 1). Now it remains to show under which conditions a matching obtained through algorithm 1 is not *CPE.*

Suppose that all sellers have at least one "*feasible buyer*", i.e., for every seller s_j there is at least one buyer b_i such that $(1-\psi)(\overline{q} - c_j) \geq u_i + \tau$. This hypothesis is without loss of generality, as the platform would never have incentives to match sellers who do not have a feasible counterpart with whom they would be willing to transact. So adding or removing those sellers from the model would not affect the platform's optimal choices or profitability. Given this hypothesis, we have that, a necessary and sufficient condition for the matching obtained through algorithm 1 to be constrained Pareto *inefficient* is that there is at least one seller such that, there is a buyer with the same index as the seller, and this buyer is *not* willing to purchase from this seller; or there are more sellers than buyers. The hypothesis that there is at least one seller who is not willing to transact with a buyer that shares the seller's index can be interpreted as requiring that the "demand crosses the supply curve".

Assumption 1. *(Each seller has at least one feasible buyer) Throughout the remainder of the article, assume, without loss of generality, that for every seller* s_j*, there is at least one buyer* b_i *such that* $(1-\psi)(\overline{q}-c_j) \geq u_i + \tau$.

Corollary 3. *(The platform's optimal matching is not CPE iff "demand crosses supply" or there are more sellers than buyers) For each seller* $s_j \in S$ *let*

$$F(s_j, \psi, \tau) \equiv \{b_i \in B; (1-\psi)(q_{ij} - c_j) \geq u_i + \tau\},$$

i.e., $F(s_j, \psi, \tau)$ *is the set of buyers, who, if matched with seller* s_j*, would end up purchasing their product. Without loss of generality, suppose that* $F(s_j, \psi, \tau) \neq \emptyset$ *for all* $s_j \in S$ *(i.e., all sellers have at least one "feasible buyer"). Suppose that* $q_{ij} = \overline{q} \in \mathbb{R}_+$ *for all* $i \leq n$ *and all* $j \leq m$*. In addition, suppose that* $c_1 < c_m$*. Then the matching obtained through algorithm 1 will not be CPE if and only if one of the following conditions holds:*

I) ("Demand crosses supply"): There is at least one $s_j \in S$*, with* $j < |B|$ *such that* $u_j \notin F(s_j, \psi, \tau)$*, or*

II) (There are more sellers than buyers): $|S| > |B|$*.*

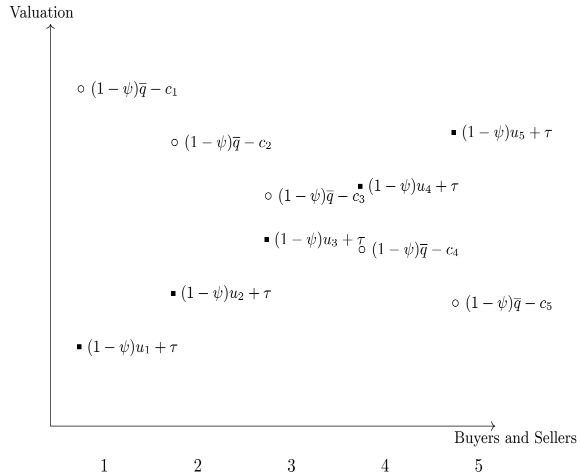
Proof: In the Appendix. ∎

Figure 3 presents instances in which the conditions from corollary 3 hold, whereas figure 4 presents instances in which those conditions are not met. Intuitively, if the "supply curve crosses the demand", implementing *CPE* matchings such as *PAM* would result in sellers with high costs remaining unmatched (because of proposition 1). But the platform could be made better off by giving away the customers who were matched to low cost sellers to the unmatched sellers with high costs, as that would increase the prices paid by these consumers, and therefore, the commissions received by the platform. Doing this replacement would also free the low cost sellers to potentially form matches with buyers with high outside options, thus increasing the number of transactions, and therefore, the fees collected by the platform. Similarly, in a market with excess demand, most sellers will be able to transact regardless of how they are matched with the "most desirable" buyers (i.e., the buyers with lowest outside options), so the platform is less likely to have incentives to implement an inefficient match. But in a market with excess supply, the platform has to choose carefully how to match sellers, as it can no longer get a transaction from all of them. In this case, the platform will have incentives to prioritize matching the inefficient sellers first (i.e., the ones with higher opportunity costs) in order to inflate equilibrium prices.

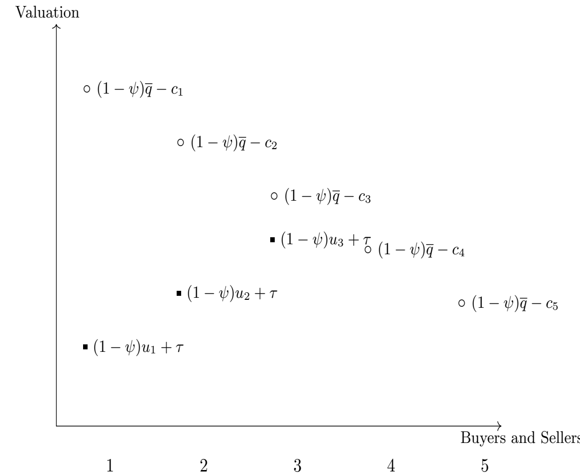
Notice that, for markets with limited supply, the hypothesis that demand crosses supply does seem very mild. Take for instance real-estate platforms such as Realtor.com or Redfin.com. In this market, even if we restricted our attention to houses that had at least one feasible buyer, try it as it may, the platform would probably not be able to get a transaction from every single listing. If this is true under the platform's optimal matching, this is also true when the platform implements PAM, which would imply that the condition that demand crosses supply holds.

When it comes to the transaction of digital goods, such as apps or movies from streaming platforms, supply always exceeds demand, as each seller can sell virtually an infinite number of units of its product.[8] So, in these instances the platform is expected to implement a matching that is not CPE as a means to inflate market prices.

[8]Notice that a market of digital goods can be interpret as one in which the supply from each product equals to the number of customers in the market (i.e., each app is subdivided into n identical products, each of which can be sold to a different customer).

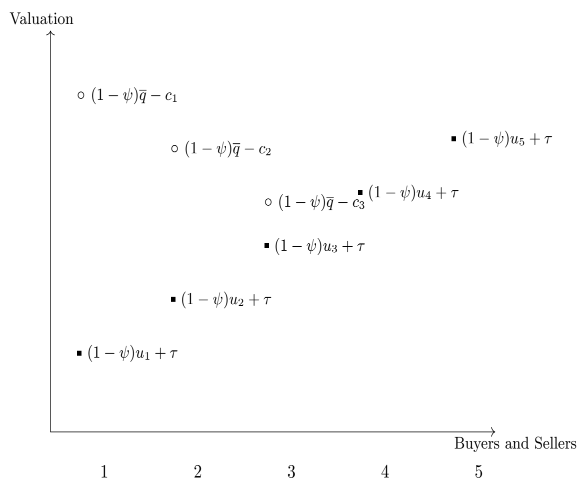


(a) A case in which "*demand crosses supply*".

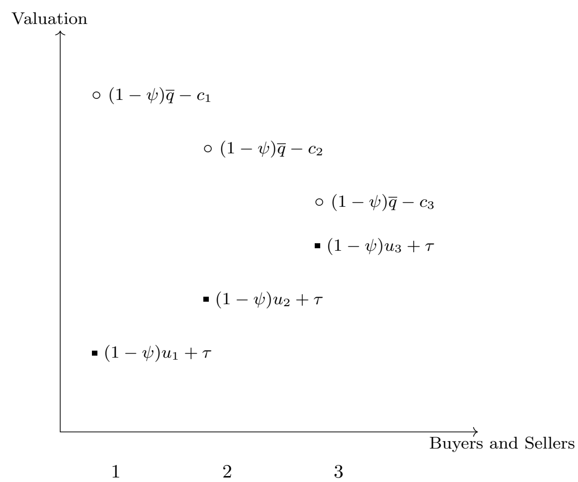


(b) A case with more sellers than buyers.

Figure 3: Instances in which the profit-maximizing matching obtained through algorithm 1 is *not CPE*.



(a) A case with more buyers than sellers, and the "*demand does not cross the supply*".



(b) A case in which there are as many buyers as sellers but the "demand does not cross the supply".

Figure 4: Instances in which the profit-maximizing matching obtained through algorithm 1 *is CPE*.

So, the results presented so far are indicative that, for many practical applications, the platform has incentives to favor matches that are **not** CPE. Our results have relied, however, on the hypothesis that products have homogeneous quality. In the following section we show how some of these results can be extended to the case in which products are vertically differentiated.

4.2 Vertically differentiated products

Throughout this section we assume that $q_{ij} = q_j \in \mathbb{R}_+$ for all $i \leq n$ and all $j \leq m$. Moreover, instead of ordering sellers in ascending order of their cost, we order sellers in descending order of the their "*net* *quality*", i.e., we assume that

$$(1-\psi)q_1 - c_1 \geq (1-\psi)q_2 - c_2 \geq \cdots \geq (1-\psi)q_m - c_m. \tag{8}$$

In this case, *PAM* would be defined as matching buyer b_1 with seller s_1, buyer b_2 with seller s_2, and so on. With this definition, one can trivially extend the results from proposition 1 and corollaries 1 and 2, to the case in which products are vertically differentiated.

Proposition 2. *(Necessary and sufficient condition for μ to be CPE) Suppose that $q_{ij} = q_j \in \mathbb{R}_+$ for all $i \leq n$ and all $j \leq m$. A matching μ is Constrained Pareto Efficient (CPE) if and only if each seller s_j such that $j \leq \min\{n,m\}$ and $(1-\psi)(q_j - c_j) \geq u_j + \tau$ is matched to a buyer b_i such that $(1-\psi)(q_i - c_i) \geq u_i + \tau$.*

Proof: Analogous to the proof of proposition 1. ■

Corollary 4. *(PAM is CPE) Suppose that $q_{ij} = q_j \in \mathbb{R}_+$ for all $i \leq n$ and all $j \leq m$. If $\mu(s_i) = b_i$ for all $i \leq \min\{n,m\}$, then μ is CPE.*

Corollary 5. *(CPE matchings maximize the number of transactions subject to a constraint) Suppose that $q_{ij} = q_j \in \mathbb{R}_+$ for all $i \leq n$ and all $j \leq m$. If a matching μ is CPE, then*

$$\mu \in \arg\max_{\mu'} |FM_{\mu'}|$$
$$s.t. \quad (1-\psi)(q_j - u_{i'}) \geq c_j + \tau \quad and \quad (1-\psi)(q_j - u_i) \geq c_{j'} + \tau \quad \forall (s_j, b_i), (s_{j'}, b_{i'}) \in FM_{\mu'}$$

Proof: Analogous to the proof of corollary 2. ■

Intuitively, proposition 2 and corollaries 4 and 5 hold because, with the hypothesis that the inequalities from (8) hold, sellers with lower index have both more feasible matches and generate more surplus in the event they transact.[9] Therefore, to achieve efficiency, we should ensure that the sellers with lower indexes are the ones (if any) who transact. Similarly, because buyers with lower index generate more surplus and have more feasible matches, efficiency requires that they are the buyers (if any) who transact. So, any matching that ensures that the buyers and sellers with lower indexes are the ones who transact, such as *PAM*, will maximize the total surplus from buyers and sellers.

As shown in the previous section, *CPE* matchings (such as *PAM*) are not necessarily profit maximizing. But one can find instances in which the platform has incentives to implement a *CPE* matching. This happens if $\tau = 0$ and either: 1) all sellers have zero opportunity cost (i.e., if $c_j = 0$ for all $j \in \{1, 2, \cdots, m\}$), or 2) all sellers have the same opportunity cost (i.e., if $c_j = c \in \mathbb{R}$ for all $j \in \{1, 2, \cdots, m\}$), and they have all the bargaining power (i.e., $\alpha = 0$). Indeed, in the former case, the revenue the platform gets from a transaction between seller s_j and buyer b_i is given by

$$\psi(1-\alpha)(q_j - u_i), \tag{9}$$

[9]To see this, notice that a transaction between buyer b_i and seller s_j occurs if and only if

$$(1-\psi)q_j - c_j \geq u_i + \tau,$$

so, the higher $(1-\psi)q_j - c_j$ is, the more feasible matches seller s_j has. In addition, the surplus that a seller s_j generates when he transacts with a buyer b_i is given by

$$q_j - \psi p_j^*(\psi, \tau, b_i) - c_j - u_i - \tau = \frac{1}{1-\psi}[1-(1-\alpha)\psi]\left[(1-\psi)q_j - c_j - (1-\psi)u_i - \tau\right],$$

which is increasing in $(1-\psi)q_j - c_j$.

whereas in the latter case, this expression reduces to

$$\psi(q_j - u_i),$$

so that in both cases the platform's revenues are proportional to the sum of value created from transactions.

Proposition 3. *(Sufficient conditions for PAM to be profit-maximizing) Suppose that $\tau = 0$, and suppose that either $c_j = 0$ for all $j \in \{1, 2, \cdots, m\}$, or $\alpha = 0$ and $c_j = c \in \mathbb{R}_+$ for all $j \in \{1, 2, \cdots, m\}$. Then, for any given $\psi \in [0, 1]$, PAM maximizes the profits of the platform.*

Proof: Proof in the Appendix. ■

Intuitively, the first part of proposition 3 holds because, if costs are zero and $\tau = 0$, the fee ψ charged by the platform is non-distortionary (i.e., it does not affect agents' decisions), in which case maximizing the creation of value allows the platform to extract more surplus. In practice, we could have $c_j = 0$ for all sellers if costs are sunk and sellers have no other means of selling their products but through the platform (e.g., sellers trying to resell tickets for concerts or sport events that they can no longer attend to, and having no other means of reselling their tickets).

Similarly, if all the sellers have the same opportunity cost, the ones with high quality products are also the ones who are capable of generating more revenues to the platform. So the platform must ensure those sellers end up transacting in equilibrium (i.e., that the "most efficient" sellers end up transacting). Moreover, $\tau = 0$ and $\alpha = 0$ removes the platform's incentives to generate more transactions than the socially optimum, as it causes profits to be proportional to the total surplus created from transactions.

For most practical applications, however, we do not expect the hypotheses from proposition 3 to hold, so that *PAM* will usually not be profit-maximizing. An alternative matching that the platform may consider is the one obtained through algorithm 1. But as shown in the Appendix, this algorithm too is not necessarily profit-maximizing when products are vertically differentiated. Indeed, when products are homogeneous, it makes sense for the platform to prioritize finding good matches to sellers with a higher index, as they are the ones who generate more revenues in the event they transact and they are also the ones with products that are "*harder to sell*" (i.e., have fewer feasible matches). The same is not necessarily true, however, when products are vertically differentiated. Indeed, if the inequalities from (8) hold, sellers with a higher index have products that are "*harder to sell*" but they are not necessarily the ones who generate more revenues to the platform in the event they transact, as they may have lower costs which may cause them to charge lower prices in equilibrium. If, however, we also assumed that

$$(1-\alpha)q_1 + \alpha\frac{c_1}{1-\psi} \leq (1-\alpha)q_2 + \alpha\frac{c_2}{1-\psi} \leq \cdots \leq (1-\alpha)q_m + \alpha\frac{c_m}{1-\psi},$$

then sellers with a higher index would also be the ones with a higher propensity to charge higher prices, in which case it would make sense for the platform to prioritize finding good matches to these sellers, which can be accomplished by following algorithm 1.

Proposition 4. *(Sufficient conditions for algorithm 1 to maximize profits and number of transactions) Suppose that $q_{ij} = q_j \in \mathbb{R}_+$ for all $i \leq n$ and all $j \leq m$. Also suppose that sellers with a higher index charge higher prices in the event they transact:*

$$(1-\alpha)q_1 + \alpha\frac{c_1}{1-\psi} \leq (1-\alpha)q_2 + \alpha\frac{c_2}{1-\psi} \leq \cdots \leq (1-\alpha)q_m + \alpha\frac{c_m}{1-\psi}.$$

Then, given $\psi \in [0,1)$ and $\tau \geq 0$, the matching allocation obtained from algorithm 1:

i) Maximizes the number of transactions that can be generated in the economy, i.e., it maximizes $|S_\mu|$.

ii) Maximizes the profits of the platform, i.e., it solves the maximization problem (5)*.*

Proof: The proof is analogous to the proof of theorem 1, as both proofs only rely on the fact that $p_j(s_i,\psi,\tau) \geq p_{j'}(s_i,\psi,\tau)$ for all $j \geq j'$ (i.e., sellers with higher index charge higher prices in the event they transact) and $F(s_j,B,\psi,\tau) \subseteq F(s_{j'},B,\psi,\tau)$ for all $j \geq j'$ (i.e., sellers with higher index have less feasible matches). ■

Intuitively, proposition 4 holds because equilibrium prices and number of transactions are maximized when the platform prioritizes finding feasible matches to sellers who face lower demand and who, simultaneously, are more likely to charge higher prices in equilibrium.

Another instance in which algorithm 1 can be used to maximize the platform's revenues is when the platform only charges a flat fee $\tau > 0$ and sets $\psi = 0$. Indeed, in this case, the platform's sole objective is to maximize the total number of transactions, and the maximization of total number of transactions can be achieved if the platform prioritizes securing good matches to sellers who have less feasible buyers.

Proposition 5. *(Algorithm 1 maximizes profits when $\psi = 0$) Suppose that $q_{ij} = q_j \in \mathbb{R}_+$ for all $i \leq n$ and all $j \leq m$. Then, for a given $\tau > 0$ and $\psi = 0$, the matching allocation obtained from algorithm 1 maximizes the number of transactions that can be generated in the economy, i.e., it maximizes $|S_\mu|$, which implies that it maximizes the profits of the platform (i.e., it solves the maximization problem* (5)*).*

Proof: The proof is analogous to the first part of the proof of theorem 1, as both proofs only rely on the fact that $F(s_j,B,\psi,\tau) \subseteq F(s_{j'},B,\psi,\tau)$ for all $j \geq j'$ (i.e., sellers with higher index have less feasible matches). ■

In practice, however, the conditions for propositions 4 and 5 to hold are rather strong. Indeed, proposition 4 requires that more expensive products are also the ones with lower demand, a condition that does not necessarily hold in practical applications. As to proposition 5, it requires that the platform only charges a flat fee per transaction and sets $\psi = 0$, which is inconsistent with the commission pricing criteria adopted in most two-sided markets. Moreover, both results do not apply to cases in which products are horizontally differentiated. In the next section we show that, for any configuration of products' quality and costs, the platform can implement the *Hungarian method*, instead, to either find a *CPE* matching or a profit-maximizing one.

4.3 Horizontally differentiated products

When products are horizontally differentiated, it is no longer clear what Positive Assortative Matching means, as now each seller s_j has a vector of attributes $(q_{1j}, q_{2j}, \cdots, q_{nj})$, so that it is no longer clear how they should be ordered. If we were to ignore those attributes and simply ordered sellers in ascending order of their costs, i.e., if we were to assume that

$$c_1 \leq c_2 \leq \cdots \leq c_m,$$

and defined *PAM* as matching buyer b_1 with seller s_1, buyer b_2 with seller s_2, and so on; then *PAM* would not necessarily be *CPE* (see example 3, from the Appendix). This is because, under product differentiation, the sellers with lowest costs are not necessarily the "most efficient ones", as they may also have an average lower quality. Similarly, algorithm 1 is not guaranteed to generate a profit-maximizing outcome (see example 3 from the Appendix).

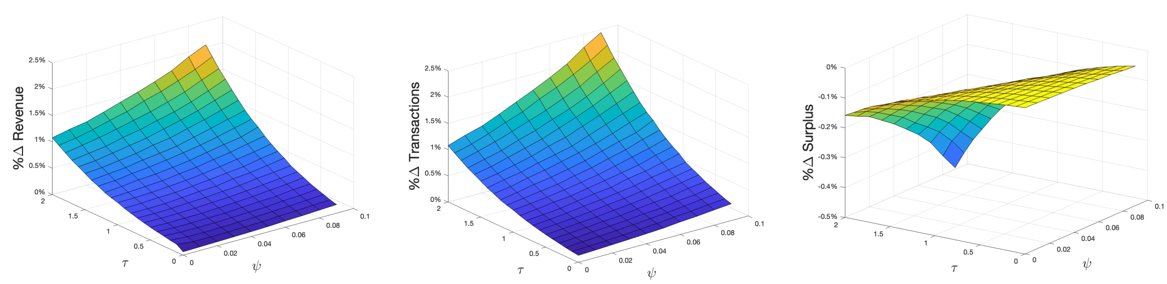
But under horizontal differentiation the platform can rely on the *Hungarian method*, instead, to find a *CPE* matching, as well as a profit-maximizing matching in polynomial time. The Hungarian method is a combinatorial optimization algorithm that can be used to minimize total cost from assigning a group of agents to a mutually exclusive task: in our case, we are interested in assigning each seller to a different buyer (or keep the seller unassigned if there are more sellers than buyers). Details on how this algorithm works can be found in section C from the Appendix. Through this method, we are able to simulate the differences in total revenue and total number of transactions obtained when the platform implements a *CPE* matching vs. a profit-maximizing one.

Our simulations indicate that *CPE* matchings can be much less profitable than the platform's optimum policy. Moreover, consistent with the theoretical results obtained for the case in which products are homogeneous, profit-maximizing matchings usually generate more transactions than *CPE* ones.

In the following simulations, we assume that costs and outside options are iid, and follow a normal distribution. More precisely, we assume that $c_j \sim N(10,1)$ and $u_i \sim N(15,1)$ for all $j \leq m$ and all $i \leq n$. Quality is proportional to costs, and we assume that

$$q_{ij} = \xi_{ij} + \beta c_j,$$

where $\xi_{ij} \sim N(0,1)$ is an iid shock, and $\beta \geq 0$ is a parameter measuring how quality correlates with production costs. To implement the simulations using the Hungarian method, we used the *R* package "*RcppHungarian*" (Silverman et al. (2022)).



(a) Difference in revenues between a proft-maximizing and a *CPE* matching

(b) Difference in number of transactions between a proft-maximizing and a *CPE* matching

(c) Difference in Buyer and Seller surplus between a proft-maximizing and a *CPE* matching

Figure 5: Plots depicting the average percentage difference in profits, number of transactions and surplus between a profit-maximizing and a *CPE* matching for different levels of ψ and τ, assuming that $\beta = 3$ and that $n = m = 20$. In total, $100,000$ simulations were conducted for each combination of ψ and τ from the grid. The percentage differences were computed by taking the difference between the average of the variable of interest from the profit maximizing matching and the one from the *CPE* matching, and dividing this difference by the average obtained from the *CPE* matching.

Figure 5 depicts the percentage difference between the average revenues, average number of transactions, and average surplus from buyers and sellers when implementing a profit-maximizing matching vs. a *CPE* one, when $\beta = 3$ and there are exactly 20 buyers and 20 sellers in the market. As the figure illustrates, these differences can be quite significant.

5 Stability

In this section we analyze the performance of *CPE* matchings in terms of *stability*. Our notion of stability is similar to the one studied in benchmark models of matching theory (e.g., Roth and Sotomayor (1989)), with the difference that in our case matched agents can negotiate their division of surplus (such as in Kelso and Crawford (1982)), and they must pay a search cost $r \geq 0$ to form a blocking coalition. Stability is a desirable property in matching markets, as it implies agents have incentives to follow through the recommendations made by the matchmaker (Roth and Sotomayor (1990)), in our case, the platform. Moreover, in our environment, when agents do not follow the platform's recommendations, the overall number of transactions can diminish (see example 4), thus reducing the profits from the platform. We show that *CPE* matches are always *stable* in the sense that no buyer is willing to pay the search cost $r \geq 0$ to form a *blocking coalition*. Meanwhile, the optimal matching allocation for the platform obtained by following algorithm 1 is not always stable. Because *CPE* are the only stable matches when r is sufficiently low, this result indicates that the platform will have more incentives to implement a *CPE* match if it believes agents can easily find matches better than the ones recommended by the platform, e.g., by scrolling down the page of search results, or by doing a more refined search.

Suppose that the platform charges commission fees (ψ, τ) and implements the matching μ. Suppose that, after the platform implements this matching, each buyer $b_i \in B$ can pay a search cost $r \geq 0$ to be matched with a different seller in the market. If, during this second round, buyer b_i chooses to pay r to be matched with a seller $s_j \neq \mu(s_j)$, seller s_j will then choose whether to transact with its current match $\mu(s_j)$, or transact with buyer b_i.

Similar to the previous section, we will also assume that sellers are ordered according to their "net quality" as follows

$$(1-\psi)q_1 - c_1 \geq (1-\psi)q_2 - c_2 \geq \cdots \geq (1-\psi)q_m - c_m.$$

In this case, *CPE* matches are obtained by matching each seller s_j such that $j \leq \min\{n, m\}$ and $(1-\psi)(q_j - c_j) \geq u_j + \tau$ with a buyer b_i such that $i \leq \min\{n, m\}$ and $(1-\psi)(q_i - c_i) \geq u_i + \tau$ (see proposition 2).

We say a matching μ is *stable* if no buyer has incentives to search for a match different than the one recommended by the platform.

Definition 3. *(Stability) Given $\psi \geq 0$ and $\tau \geq 0$, and given the search cost r, a matching μ is stable if, for every $s_j \in S$ there is a payoff $w_{s_j} \in \mathbb{R}$ and for every $b_i \in B$ there is a payoff $w_{b_i} \in \mathbb{R}$ such that:*

i) (Individual rationality) $w_{s_j} \geq c_j$ $\forall s_j \in S$ and $w_{b_i} \geq u_i$ $\forall b_i \in B$ (i.e., agents' payoffs are greater than or equal to the their outside option).

ii) (Feasibility) If $\mu(s_j) = b_i \in B$ and $w_{b_i} > u_i$ or $w_{s_j} > c_j$, then we must have $q_j - u_i > \frac{c_j+\tau}{1-\psi}$ and there is a $p_j \in \left[\frac{c_j+\tau}{1-\psi}, q_j - u_i\right]$ such that $w_{s_j} = (1-\psi)p_j - \tau$ and $w_{b_i} = q_j - p_j$. If $\mu(s_j) = s_j$, then $w_{s_j} = c_j$, and if $\mu(b_i) = b_i$, then $w_{b_i} = u_i$ (i.e., unmatched agents get their outside option).

iii) (No Blocking Coalition) There is no $(s_{j'}, b_{i'}) \in S \times B$ and no $(\tilde{w}_{s_{j'}}, \tilde{w}_{b_{i'}}, \tilde{p}) \in \mathbb{R}^3$ such that:

$$q_{j'} - u_{i'} - r > \frac{c_{j'} + \tau}{1-\psi},$$

with

$$\tilde{p} \in \left[\frac{c_{j'} + \tau}{1-\psi}, q_{j'} - u_{i'} - r\right],$$

and

$$\tilde{w}_{s_{j'}} = (1-\psi)\tilde{p} - \tau > w_{s_{j'}},$$

$$\tilde{w}_{b_{i'}} = q_{j'} - \tilde{p} - r > w_{s_{j'}}.$$

Clearly, if the search cost r is sufficiently high, every matching is stable. In this case the platform does not need to worry about implementing a matching that incentivizes agents to follow through its recommendations.

Theorem 2. *If*

$$(1-\psi)q_1 - c_1 \geq (1-\psi)q_2 - c_2 \geq \cdots \geq (1-\psi)q_m - c_m,$$

then,

i) Any CPE matching is stable.

ii) If $r = 0$ a matching is stable if and only if it is CPE.

iii) If

$$(1-\psi)r \geq \max_j\{(1-\psi)q_j - c_j\} - (1-\psi)\min_i\{u_i\} - \tau, \tag{10}$$

any matching μ is stable.

Proof: In the Appendix. ■

Theorem 2 states that stability is more easily attained if a *CPE* matching is implemented. Intuitively, theorem 2 is similar to a known result in benchmark matching models without transaction or search costs, and in which agents' payoffs from a match are pre-determined and non-negotiable. The result states that when both sides of the two-sided market have the same ordinal preferences over agents on the other side of the market, and those preferences are strict, *PAM* is the only stable matching allocation (e.g., see Gusfield and Irving (1989)). However, in our environment, it is not immediately clear which sellers are the ones most preferred by buyers and vice versa, as agents must engage in a bargaining process after the match is determined. But notice that, under the commission fee $\psi \geq 0$ and $\tau \geq 0$, sellers with the highest $(1-\psi)q_j - c_j$ are the ones who can generate most surplus to be shared with buyers, so they can be interpreted as the "most preferred sellers". Similarly, buyers with lower u_i are the ones who can generate most surplus to be shared with sellers, so they can be interpreted as the "most preferred buyers". So, following this intuition, in our model with search and transaction costs, a stable matching is more likely to be achieved if the platform matches the "most desirable seller" (i.e., the one with highest $(1-\psi)q_j - c_j$) with the "most desirable buyer" (i.e., the one with lowest u_i), the "second most desirable seller" (i.e., the one with second highest $(1-\psi)q_j - c_j$) with the "second most desirable buyer" (i.e., the one with second lowest u_i), and so on.

This result implies that policies aimed towards reducing r, such as prohibiting platforms from obfuscating search results (e.g., by prohibiting platforms from hiding shipping costs) can have a positive effect on consumer and seller surplus, by giving the platform more incentives to implement a *CPE* matching such as *PAM*.

The next example illustrates how instability may reduce the overall profits of the platform.

Example 2. ***(To guarantee stability, the platform may have incentives to implement PAM)*** *Consider a market where the set of sellers is given by* $\{s_1, s_2, s_3, s_4\}$*, and the set of buyers is given by* $\{b_1, b_2, b_3, b_4\}$*. Sellers' quality and costs are given by*

$$(q_1, q_2, q_3, q_4) = (20, 20, 20, 20) \quad and \quad (c_1, c_2, c_3, c_4) = (2, 8, 14, 16),$$

respectively. Customers' outside option are given by

$$(u_1, u_2, u_3, u_4) = (0, 3, 5, 7).$$

If PAM is implemented, then

$$s_1 : b_1 \quad s_2 : b_2, \quad s_3 : b_3, \quad s_4 : b_4$$

If the platform charges commission fees $\psi = 0$ *and* $\tau = 1$*, then this Pareto efficient match would generate a total of* 3 *transactions, resulting in a total profit of* $3 \times \tau = 3$ *to the platform.*

But if it implemented the match

$$s_1 : b_4, \quad s_2 : b_3, \quad s_3 : b_2, \quad s_4 : b_1 \tag{11}$$

then 4 transactions would take place, so the platform's profits would equal to $4 \times \tau = 4$*.*

However, if the search cost is given by $r = 1$*, it can be shown that the match* (11) *is unstable, as buyer* b_1 *would have incentives to form a blocking coalition with seller* s_1*, and buyer* b_2 *would have incentives to form a blocking coalition with seller* s_3*. Once these blocking coalitions are formed, buyers* b_3 *and* b_4 *remain unmatched. If they decided to pay the search cost* $r = 1$ *to be matched with sellers* s_3 *or* s_4*, they would end up with a payoff lower than their outside option, so they have no incentives to try to form a different match through the platform. As a result, in the end only two transactions take place, generating a profit of* $2 \times \tau = 2$ *to the platform.*

Meanwhile, PAM is stable (theorem 2). So the platform may be better off implementing PAM and getting a payoff of 3 *as opposed to implementing the matching* (11) *and getting a payoff of* 2*.*

When $\psi = 0$ we can derive a qualitative result similar to theorem 2 for cases in which transacting outside the platform is feasible and is the only way agents can circumvent the recommendations made by the platform. In fact, in this case, the negative effects of instability to the platform tend to be higher, as instability would result in the platform not being able to collect fees from transactions that bypassed its recommendation (see section E from the appendix for a thorough analysis).

6 Discussion

We build a simple theoretical model that sheds some light into two-sided platforms' incentives to steer customers towards certain products based on their willingness to pay. We show that the platform will tend to have incentives to induce more transactions than the socially optimum. So this shows that, even if a platform only sells third-party products, so that it cannot engage in self-preferencing, the platform may still have incentives to implement a recommendation system that harms buyers and sellers.

We also derive some conditions under which the platform will have more incentives to adopt this type of recommendation that harms buyers and sellers. In particular, the higher the search costs, the more incentives the platform has to generate this excessive number of transactions, as in this case matchings that maximize the number of transactions will be more likely to be stable. This implies that policies directed towards reducing search obfuscation (e.g., by prohibiting platforms from hiding add-on prices, such as shipping costs), can have a positive impact on sellers' and customers' surplus.

While our environment is not dynamic, we partially capture the platform's reputation concerns by providing conditions under which a matching is stable. Indeed, if a matching is not stable, customers and sellers systematically receive "bad matches" in the sense that they are able to form better incentive-compatible matches on their own. So one could argue that instability reduces the probability that someone revisits the platform for future purchases.

This work has some limitations that can be explored by future research. Perhaps one of its biggest limitations is that, like in Hagiu and Jullien (2011) and Boerner and Quint (2023), we do not characterize the equilibrium when prices are (endogenously) set *before* the matchmaking takes place, a case that would be more consistent with many prominent platforms, such as Amazon.com. This restriction is mainly imposed for tractability to allow us to solve the model analytically. Future research could explore the effects of relaxing this restriction on sellers' optimal pricing behavior. The mechanics presented in this article may already shed some light on what might happen if prices chosen by sellers could affect search results: on the one hand, charging a lower price increases the probability that the product is sold, which would give the platform more incentives to match buyers with sellers who charge lower prices; but on the other hand, a lower price generates less revenue to the platform, in which case the platform would have incentives to give a higher priority on search results to more expensive products. If these two effects cancel each other out, it could eliminate sellers' incentives to offer large discounts as a means to improve their ranking on search results, thus harming consumer welfare.

Another limitation from this study is that we consider a scenario in which the platform knows agents' exact characteristics, when in most practical applications the platform only has an imprecise estimate of products' quality and agents' willingness to buy and sell. Future research may consider the effects that imprecise information on agents' types may have on the platform's optimal policy. Perhaps, if types are sufficiently imprecise, the platform may find it preferable to implement some degree of positive assortative matching in order to maximize the probability that matched buyers and sellers actually transact.

A Proofs

Proof of proposition 1:

Necessity: Consider a customer b_i such that $i \leq \min\{n, m\}$ and $(1-\psi)(\overline{q} - c_i) \geq u_i + \tau$, and define $\overline{j} = \max_j\{j \in \mathbb{N}; j \leq \min\{m, n\}$ and $(1-\psi)(\overline{q} - u_j) \geq c_j + \tau\}$.

(a) Suppose by way of contradiction that customer b_i is matched with a seller s_j such that $j > \overline{j}$. Then there must be a seller s_l such that $l \leq \overline{j}$, that is either unmatched or is matched with a buyer b_k such that $k > \overline{j}$. If seller s_l is unmatched, the allocation is clearly inefficient, as total buyer and seller surplus would be increased if buyer b_i was matched with the idle seller s_l instead of being matched with seller s_j, as seller s_l has a lower cost (and the same quality). If, on the other hand, seller s_l is matched with a buyer b_k such that $k > \overline{j}$, the sum of the value created by the match between buyer b_i and seller s_j and the match between buyer b_k and seller s_l is given by

$$
\begin{aligned}
&[\overline{q} - c_j - u_i - \tau - \psi p_j(b_i, \psi, \tau)] + [\overline{q} - c_l - u_k - \tau - \psi p_l(b_k, \psi, \tau)] \\
&= [\overline{q} - c_j - u_k - \tau - \psi p_j(b_k, \psi, \tau)] + [\overline{q} - c_l - u_i - \tau - \psi p_l(b_i, \psi, \tau)] \,, \qquad (12)
\end{aligned}
$$

assuming that both matches generate a transaction (otherwise, the proof is trivial). If $k > j$, then $u_k > u_j$, which implies that

$$[\overline{q} - c_j - u_k - \tau - \psi p_j(b_k, \psi, \tau)] < [\overline{q} - c_j - u_j - \tau - \psi p_j(b_j, \psi, \tau)] < 0,$$

where the last equality comes from the fact that $(1-\psi)(\overline{q} - u_j) < c_j + \tau$. If $k \leq j$, then

$$[\overline{q} - c_j - u_k - \tau - \psi p_j(b_k, \psi, \tau)] < [\overline{q} - c_k - u_k - \tau - \psi p_k(b_k, \psi, \tau)] < 0,$$

where the last equality comes from the assumption that $(1-\psi)(q_k - u_k) < c_k + \tau$. Either way, we must have that $[\overline{q} - c_j - u_k - \tau - \psi p_j(b_k, \psi, \tau)] < 0$, which implies that the total customer and seller surplus created from these two matches (expression (12)) is less than or equal to

$$[\overline{q} - c_l - u_i - \tau - \psi p_l(b_i, \psi, \tau)] \,,$$

the buyer and seller surplus created if seller s_l is matched to buyer b_i, and seller s_j and customer b_k remain unmatched (or matched with one another generating a surplus of zero). Because a new allocation can be found that increases the total customer and seller surplus, we conclude that the original allocation is not *CPE*.

(b) Suppose by way of contradiction that buyer b_i is unmatched. Then there must be a seller s_l with $l \leq \overline{j}$ that is either unmatched or is matched to a buyer b_k such that $k > \overline{j}$. If seller s_l is unmatched, the allocation is clearly inefficient, as total buyer and seller surplus would be increased if buyer b_i was matched with the idle seller s_l instead of being unmatched.

Analogously, if seller s_l is matched with a buyer b_k such that $k > \overline{j}$, then, the buyer and seller surplus from this match is given by

$$\max\{\overline{q} - c_l - u_k - \tau - \psi p_l(b_k, \psi, \tau), 0\},$$

which is less than $\overline{q} - c_l - u_i - \tau - \psi p_l(b_i, \psi, \tau)$, the buyer and seller surplus created if seller s_l was matched with buyer b_i instead, and buyer b_j remained unmatched.

In either of these contingencies, an alternative allocation can be found that improves total buyer and seller surplus, which implies that an allocation in which buyer b_i remains unmatched cannot be Pareto efficient.

Sufficiency: Because there is a finite number of possible matchings, there must be at least one that maximizes total buyer and seller surplus. Because we saw that a necessary condition for the maximization of surplus is that all buyers b_i such that $i \leq \overline{j}$ are matched with sellers s_j such that $j \leq \overline{j}$, and because the total buyer and seller surplus from any such matching is given by

$$\sum_{j \leq \bar{j}} \left[\overline{q} - c_j - u_j - \tau - \psi p_j(b_j, \psi, \tau)\right],$$

we conclude that these matchings maximize the total buyer and seller surplus.

■

Proof of corollary 2:
Suppose that

$$\mu \in \arg\max_{\mu'} |FM_{\mu'}|$$
$$s.t. \qquad (1-\psi)(\overline{q} - u_{i'}) \geq c_j + \tau \quad \text{and} \quad (1-\psi)(\overline{q} - u_i) \geq c_{j'} + \tau \quad \forall (s_j, b_i), (s_{j'}, b_{i'}) \in FM_{\mu'} \tag{13}$$

and let μ' be a Constrained Pareto efficient (CPE) matching. From proposition 1, we have that every CPE matching satisfies constraint (13). So it suffices to show that $|S_{\mu'}| \geq |S_\mu|$.

Suppose by way of contradiction that $|S_{\mu'}| < |S_\mu|$. Then there is a $s_j \in S_\mu$ with $\mu(s_j) = b_i$ such that $s_j \notin S_{\mu'} = \{s_j \in S; j \leq \min\{m, n\}$ and $(1-\psi)(\overline{q} - u_j) \geq c_j + \tau\}$. Because $s_j \notin S_{\mu'}$, we must have $j > j'$ for every $s_{j'} \in S_{\mu'}$ and every $b_{j'} \in B_{\mu'}$. If $\mu(s_j) = b_{i'} \notin B_{\mu'}$, we would have a contradiction with μ' being CPE, as in this case we would be able to create an additional match with positive surplus without affecting the other transactions conducted under μ'. So we must have $\mu(s_j) = b_{i'} \in B_{\mu'}$. This implies that, if $|S_{\mu'}| < |S_\mu|$, there must be a $s_{j'} \in S_{\mu'}$ such that $\mu(s_{j'}) = b_i \in B_\mu$, with $b_i \notin B_{\mu'}$.

But in this case, if $j > i$, then

$$c_j + \tau \geq c_i + \tau > (1-\psi)(\overline{q} - u_i),$$

a contradiction with constraint (13) being satisfied for the matching μ, since in this case seller s_j would *not* be willing to transact with customer b_i, even though both s_j and b_i transact under μ.

So suppose that $j \leq i$. In this case, we have that

$$(1-\psi)(\overline{q} - u_j) < c_j + \tau \leq c_i + \tau,$$

which gives us the same contradiction (i.e., seller s_j and buyer b_i both transact under μ, and yet, they are not willing to transact with each other, a violation of constraint (13)). ■

Proof of proposition 3:
Suppose that $\tau = 0$.

Case 1) If $c_j = 0$ for all $j \in \{1, 2, \cdots, m\}$, then, for any given $\psi \in [0,1]$ and any matching function μ, a matched agent will transact under μ and $\psi \in [0,1]$ if and only if he transacts under μ and $\psi = 0$ (i.e., the commission ψ does not affect agents' willingness to transact). Therefore, from expression 9 we have that the platform's revenue for a given match μ is proportional to the total value created from transactions when there are no commissions. So the platform has incentives to implement a match that maximizes total surplus.

Case 2) Suppose that $\alpha = 0$ and $c_j = c$ for all $j \in \{1, 2, \cdots, m\}$. Then, defining $\overline{j} = \max_j\{j \in \mathbb{N}; j \leq \min\{m, n\} \text{ and } (1-\psi)(q_j - u_j) \geq c_j\}$, we have that the platform's profits obtained by implementing *PAM* is given by

$$\sum_{j \leq \overline{j}} \psi(q_j - u_j). \tag{14}$$

Now consider an alternative matching μ. Let $S_\mu \equiv \{s_j \in S; \mu(s_j) = b_i \in B \wedge (1-\psi)(q_j - u_i) \geq c_j\}$, i.e., S_μ is the set of sellers who end up transacting under μ. Suppose by contradiction that there is some seller s_j with $j > \overline{j}$ such that $s_j \in S_\mu$, i.e., there is some seller who does not transact under *PAM* but does transact under μ. Then $\mu(s_j) = b_i$, with $i \leq \overline{j} < j$. This implies that there is at least one seller s_l with $l \leq \overline{j}$ such that either $\mu(s_l) = s_l$ or $\mu(s_l) = c_k$, with $k > \overline{j}$. In either case, the platform's total profits can be improved if seller s_j remains unmatched and seller s_l is matched with c_i.

So in a profit maximizing match only sellers s_j with $j \leq \overline{j}$ should transact. Clearly, these sellers should be matched with the $\overline{j}$ customers with lowest outside option, so as to increase prices, and therefore the commissions paid. From equation (14) one can clearly see that any matching μ such that, for all $j \leq \overline{j}$, we have that $\mu(s_j) = b_i$ with $i \leq \overline{j}$ maximizes profits. In particular, *PAM* satisfies this property.

■

The following proofs will make use of the following definition:

Definition 4. *For a given pair of ψ and τ, we define*

$$\begin{aligned}
&\widehat{B_\mu} \equiv \{i \in \{1, 2, \cdots, n\}; \mu(b_i) = s_j \in S \wedge (1-\psi)(q_j - u_i) \geq c_j\}, \\
&\widehat{S_\mu} \equiv \{j \in \{1, 2, \cdots, m\}; \mu(s_j) = b_i \in B \wedge (1-\psi)(q_j - u_i) \geq c_j\}, \\
&B_\mu \equiv \{b_i\}_{i \in \widehat{B_\mu}}, \\
&S_\mu \equiv \{s_i\}_{i \in \widehat{S_\mu}},
\end{aligned}$$

In words, B_μ and S_μ correspond to the set of buyers and sellers, respectively, who end up transacting after the match μ is formed, given that the platform's transaction fee is ψ; whereas $\widehat{B_\mu}$ and $\widehat{S_\mu}$ correspond to the indexes of those buyers and sellers, respectively. Notice that, because the matching is one-to-one, we must have $|B_\mu| = |S_\mu|$ for any matching function μ. Also notice that these sets depend on ψ. But because ψ will be treated as given in the following proofs, we did not index these sets by ψ to avoid clutter notation.

Proof of theorem 1:

Part i)

Let μ^* be the matching obtained by implementing algorithm 1, and let μ be any other matching. Clearly, if $B_{\mu^*} = \emptyset$, then $B_\mu = \emptyset$. So suppose that $|B_{\mu^*}| > 0$.

Suppose by contradiction that $|B_\mu| > |B_{\mu^*}|$ (which happens iff $|S_\mu| > |S_{\mu^*}|$). Then there must be an $i \in \widehat{B_\mu}$ such that $i \notin \widehat{B_{\mu^*}}$. Because μ^* sequentially matches buyers with lowest outside options first (i.e., those with lower indexes), this implies that $i > |B_{\mu^*}|$. If $\mu(b_i) = s_j \notin S_{\mu^*}$, we get a contradiction. Indeed, since $\mu(b_i) = s_j \in S_\mu$, we have that buyer b_i is a *feasible* match to seller s_j. But because $i > |B_{\mu^*}|$, by the time seller s_j was selected in algorithm 1 (i.e., the algorithm used to form the matching μ^*) to form a match with one of the remaining customers, customer b_i was still available to that seller, so that $b_i \in F(s_j, B_j, \psi, \tau)$, a contradiction with $\mu^*(s_j) = s_j$.

Therefore, for each $b_i \in B_\mu$ such that $b_i \notin B_{\mu^*}$, we must have that $\mu(b_i) \in S_{\mu^*}$, i.e., if a buyer transacts under μ but not under μ^*, then this buyer must transact with one of the sellers who transacts under μ^*. Therefore, if there is a seller $s_k \in S_\mu$ such that $s_k \notin S_{\mu^*}$, there exists $b_l \in B_{\mu^*}$ such that $\mu(b_l) = s_k$. Moreover, because $|B_\mu| > |B_{\mu^*}|$, there must be a $b_i \in B_\mu \backslash B_{\mu^*}$ such that and $\mu(b_i) = s_j \in S_{\mu^*}$. But if $\mu(b_i) = s_j$ and $b_i \notin B_{\mu^*}$, then every seller s_k with $k < j$ (i.e., with $c_k \leq c_j$) must be matched under μ^*, as b_i is a feasible buyer to seller s_j (i.e., $b_i \in F(s_j, B, \psi, \tau)$), and thus, feasible to all sellers with a lower cost, and is available to be matched by the time seller s_k with $k < j$ is selected in algorithm 1.

Therefore, if $s_k \notin S_{\mu^*}$, we must have $k > j$ (i.e., $c_k \geq c_j$). Moreover, if $\mu(b_l) = s_k$, then b_l is a feasible match to seller s_k (i.e., $b_l \in F(s_k, B, \psi, \tau)$). But since $s_k \notin S_{\mu^*}$, then by the time s_k is selected to form a match in algorithm 1, buyer b_l was already matched with some other seller $s_{k'}$ such that $k' > k$ (i.e., with cost $c_{k'} \geq c_k$).

We claim that there is at least one seller s'_k with $k' > k$ such that $s_{k'} \in S_{\mu^*} \backslash S_\mu$. Indeed, for each seller $s_{k'}$ such that $k' > k$, $F(s_{k'}, B, \psi, \tau) \subseteq F(s_k, B, \psi, \tau)$. So each seller $s_{k'} \in S_{\mu^*}$ such that $k' > k$ must be matched with a buyer in $F(s_k, B, \psi, \tau)$. Because all buyers in $F(s_k, B, \psi, \tau)$ belong to B_{μ^*}, and because $\mu(s_k) = b_l \in F(s_k, B, \psi, \tau)$, one-to-one matching implies that there must be at least one $s_{k'} \in S_{\mu^*}$ that is either unmatched or matched with a buyer not in $F(s_k, B, \psi, \tau)$. In either case, such a seller does not transact.

So we conclude that for every $s_k \in S_\mu$ such that $s_k \notin S_{\mu^*}$ there is a *different* $s_{k'} \in S_{\mu^*}$ such that $s_{k'} \notin S_\mu$ (i.e., we can form a bijection that maps every $s_k \in S_\mu$ such that $s_k \notin S_{\mu^*}$ into a different $s_{k'} \in S_{\mu^*} \backslash S_\mu$ such that $s_{k'} \notin S_\mu$), so that $|S_{\mu^*}| \geq |S_\mu|$.[10] Because the matching is one-to-one, this implies that $|B_{\mu^*}| \geq |B_\mu|$, a contradiction with $|B_\mu| > |B_{\mu^*}|$.

Part ii)

For pedagogic purposes, we will prove part ii) of the theorem for the case in which all inequalities in 1 and 2 are strict. The result can be easily extended to cases in which those inequalities are not strict (see footnotes 11 and 12).

Clearly, the profit from the platform as a function of μ is given by

$$\begin{aligned}\pi(\mu) &= \psi \sum_{j \in \widehat{S_\mu}} \left[(1-\alpha)(\overline{q} - u_{\mu(j)}) + \alpha \frac{c_j + \tau}{1-\psi} \right] \\ &= |S_\mu| \psi (1-\alpha) \overline{q} - \psi (1-\alpha) \sum_{i \in \widehat{B_\mu}} u_i + \frac{\psi \alpha}{(1-\psi)} \sum_{j \in \widehat{S_\mu}} (c_j + \tau),\end{aligned}$$

where, with abuse of notation, $\mu(j) = i$ such that $\mu(s_j) = b_i$. So the total profit only depends on the sets B_μ and S_μ.

Now let μ^* be the matching allocation obtained by implementing algorithm 1, and let μ be any other matching allocation.

Clearly, if $B_{\mu^*} = \emptyset$, then we must have $B_\mu = \emptyset$, in which case the proof is trivial. So let us assume that $B_{\mu^*} \neq \emptyset$.

Suppose that $S_\mu \subsetneq S_{\mu^*}$, and let $s_j \in S_{\mu^*} \backslash S_\mu$. If $b_i \notin B_\mu$ for some $b_i \in F(s_j, B, \tau, \psi)$, the platform would be able to increase its profits by forming the match $s_j : b_i$. So suppose that $b_i \in B_\mu$ for all $b_i \in F(s_j, B, \tau, \psi)$.

[10] This part of the proof could be made more formal by applying induction, i.e., by iteratively comparing $|S_\mu|$ with $|S_{\mu^*}|$ after eliminating the pairs $(s_k, \mu(s_k))$ and $(s_{k'}, \mu^*(s_{k'}))$ from μ and μ^*, respectively.

Then, since $S_\mu \subsetneq S_{\mu^*}$, there must be a $s_{j'} \in S_{\mu^*}$ with $j' < j$ s.t. $\mu(s_j) = b_i \in F(s_j, \tau, \psi)$. But then, the platform could be made better off by replacing the match $s_{j'} : b_i$ with $s_j : b_i$, leaving seller $s_{j'}$ unmatched.

So consider the following alternative cases:

1. Suppose there is some $b_i \in B_\mu$ such that $b_i \notin B_{\mu^*}$. Because algorithm 1 iteratively matches buyers with lowest outside options first, it must be the case that there are $|B_{\mu^*}|$ buyers with outside option lower than u_i. Form the first part of the theorem, we must have $|B_{\mu^*}| \geq |B_\mu|$. So there must be at least one $b_j \in B_{\mu^*} \backslash B_\mu$ such that $u_j < u_i$.[11] Then the platform's profits obtained by implementing μ can be improved by matching seller $\mu(b_i)$ with b_j as opposed to matching $\mu(b_i)$ with b_i. So we conclude that a necessary condition for μ to maximize the profits of the platform is that $B_\mu \subseteq B_{\mu^*}$.

2. Suppose there is some $s_i \in S_\mu$ such that $s_i \notin S_{\mu^*}$. Because $F(s_j, B, \psi, \tau) \leq F(s_i, B, \psi, \tau)$ for all $j > i$, there must be at least one $j > i$ such that $s_j \in S_{\mu^*} \backslash S_\mu$. [12] Then the platform's profits obtained by implementing μ can be improved by matching buyer $\mu(s_i)$ with s_j as opposed to matching $\mu(s_i)$ with s_i. So we conclude that a necessary condition for μ to maximize the profits of the platform is that $S_\mu \subseteq S_{\mu^*}$.

Therefore, if μ maximizes the profits of the platform, we must have $B_\mu \subseteq B_{\mu^*}$ and $S_\mu \subseteq S_{\mu^*}$. But, as previously argued, if $B_\mu \subseteq B_{\mu^*}$ and $S_\mu \subseteq S_{\mu^*}$, then the profits under μ^* are weakly higher than the profits under μ. ■

Proof of corollary 3:

⇒*Sufficiency. Case 1:* Suppose that there is at least one seller $s_j \in S$, such that $j < |B|$ and $u_j \notin F(s_j, \psi, \tau)$ i.e., there is a buyer with the same index as seller s_j, and this buyer would not willing to purchase from this seller. From proposition 1, in a *CPE* matching, a seller $s_{j'}$ with $j' > j$ should not transact. In particular, seller s_m will not transact under a *CPE* matching. But because $F(s_m, \psi, \tau) \neq \emptyset$, we have that seller s_m transacts under algorithm 1, so that the matching obtained through this algorithm is not *CPE*.

Case 2: Suppose that there are more sellers than buyers. Then, from proposition 1, seller s_1 transacts under a *CPE* matching, but s_m does not. But because algorithm 1 starts by matching seller s_m, the one with lowest willingness to sell, and because $F(s_m, \psi, \tau) \neq \emptyset$, seller s_m will transact under algorithm 1.

⇐*Necessity:* Follows directly from proposition 1. ■

Proof of proposition 6:

For a given match μ, we first show that the platform's objective function is upper semicontinuous (u.s.c.) in (ψ, τ). Indeed, for each $s_j \in S$ define

$$\pi_{s_j, b_i}(\psi, \tau) \equiv \begin{cases} \tau + \psi \left[(1-\alpha)(q_j - u_i) + \alpha \frac{c_j + \tau}{1-\psi} \right], & \text{if } (1-\psi)(q_j - u_i) \geq c_j + \tau \text{ and } \psi < 1 \\ \max\{q_j - u_i, 0\}, & \text{if } \psi = 1 \text{ and } c_i = \tau = 0 \\ 0, & \text{else} \end{cases}.$$

[11] If the inequalities in 1 were not necessarily strict, then we would suppose that there is some $b_i \in B_\mu$ such that $\#\{b_j \in B_\mu; u_j = u_i\} > \#\{b_j \in B_{\mu^*}; u_j = u_i\}$, and conclude that this would imply that for each $b_{\tilde{j}} \in \{b_j \in B_\mu; u_j = u_i\}$ such that $b_{\tilde{j}} \notin \{b_j \in B_{\mu^*}; u_j = u_i\}$ there is a different $b_{j'} \in B_{\mu^*} \backslash B_\mu$ such that $u_{\tilde{j}} < u_{j'}$.

[12] If the inequalities in 2 were not necessarily strict, then we would suppose that there is some $s_i \in S_\mu$ such that $\#\{s_j \in S_\mu; c_j = c_i\} > \#\{b_j \in S_{\mu^*}; c_j = c_i\}$, and conclude that this would imply that for each $s_{\tilde{j}} \in \{s_j \in S_\mu; c_j = c_i\}$ such that $s_{\tilde{j}} \notin \{s_j \in S_{\mu^*}; c_j = c_i\}$ there is a different $s_{j'} \in S_{\mu^*} \backslash S_\mu$ such that $c_{\tilde{j}} > c_{j'}$.

In this case, defining $M_\mu \equiv \{(s_j, b_i) \in S \times B; \mu(s_j) = b_i\}$ (i.e., M_μ represents all the matches associated with μ, excluding agents who are matched with themselves), we have that the profit from the platform can be written as

$$\pi(\mu, \psi, \tau) = \sum_{(s_j, b_i) \in M_\mu} \pi_{s_j, b_i}(\psi, \tau).$$

Because the sum of u.s.c. functions is u.s.c., it suffices to show that $\pi_{s_j,b_i} : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is u.s.c. for every possible $(s_j, b_i) \in S \times B$.

Clearly, for values of (ψ, τ) such that $(1-\psi)(q_j - u_i) > c_j + \tau$ or $(1-\psi)(q_j - u_i) < c_j + \tau$, the function $\pi_{s_j,b_i}(\cdot,\cdot)$ is continuous, and therefore, u.s.c.

So suppose that (ψ, τ) is such that $(1-\psi)(q_j - u_i) = c_j + \tau$. If $q_j - u_i < 0$, the platform earns zero profits regardless of the commissions it charges, in which case $\pi_{s_j,b_i}(\cdot,\cdot)$ is continuous and therefore u.s.c. So throughout, let us assume that $q_j - u_i \geq 0$. Then, we must analyze the two possible scenarios:

1. Suppose that $\psi < 1$. Then there is a $\delta > 0$ such that $\psi + \delta < 1$.

   So suppose that
   $$||(\psi', \tau') - (\psi, \tau)|| = \sqrt{(\psi' - \psi)^2 + (\tau' - \tau)^2} < \delta.$$

   Then $\psi' - \psi < \delta \Rightarrow \psi' < \psi + \delta < 1$.

   So, for any $(\psi', \tau') \in B_\delta((\psi, \tau))$, we have that $\psi' < 1$, which also implies that

   $$\begin{aligned}
   \pi_{s_j,b_i}(\psi', \tau') - \pi_{s_j,b_i}(\psi, \tau) \leq& \tau' + \psi' \left[(1-\alpha)(q_j - u_i) + \alpha \frac{c_j + \tau'}{1-\psi'}\right] - \tau - \psi \left[(1-\alpha)(q_j - u_i) + \alpha \frac{c_j + \tau}{1-\psi}\right] \\
   \leq& \underbrace{(\tau' - \tau)}_{<\delta} + \underbrace{(\psi' - \psi)}_{<\delta}(1-\alpha)(q_j - u_i) + \alpha \left[\frac{(1-\psi)\psi'(c_j + \tau') - (1-\psi')\psi(c_j + \tau)}{(1-\psi')(1-\psi)}\right] \\
   <& \delta + \delta(1-\alpha)(q_i - u_i) + \alpha \left[\frac{(1-\psi)(\psi+\delta)(c_j + \tau + \delta) - (1-\psi-\delta)\psi(c_j + \tau)}{(1-\psi+\delta)(1-\psi)}\right].
   \end{aligned}$$

   The term

   $$\delta + \delta(1-\alpha)(q_i - u_i) + \alpha \left[\frac{(1-\psi)(\psi+\delta)(c_j + \tau + \delta) - (1-\psi-\delta)\psi(c_j + \tau)}{(1-\psi+\delta)(1-\psi)}\right]$$

   clearly converges to zero as $\delta \to 0$. This implies that, for any $\varepsilon > 0$, there is a δ sufficiently small such that, for any $(\psi', \tau') \in B_\delta((\psi', \tau'))$, $\pi_{s_j,b_i}(\psi', \tau') - \pi_{s_j,b_i}(\psi, \tau) < \varepsilon$, which implies that $\pi_{s_j,b_i}(\cdot,\cdot)$ is u.s.c. in (ψ, τ).

2. Suppose that $\psi = 1$ and $c_j = \tau = 0$. Then, for any $\varepsilon > 0$ and any $(\psi', \tau') \neq (\psi, \tau) = (1, 0)$,

   $$\pi_{s_j,b_i}(\psi', \tau') = \begin{cases} \psi'(1-\alpha)(q_j - u_i) + \alpha \frac{\tau'}{1-\psi'}, & \text{if } (1-\psi')(q_j - u_i) \geq c_j + \tau' \text{ and } \psi' < 1 \\ 0, & \text{else} \end{cases}$$

   Notice that if $q_j - u_i < 0$, then $\pi_{s_j,b_i}(\psi', \tau') = 0$ for all $(\psi', \tau') \in [0,1] \times \mathbb{R}_+$, in which case $\psi(\cdot,\cdot)$ is continuous, and therefore u.s.c. So suppose that $q_j - u_i \geq 0$. Then, for any $(\psi', \tau') \neq (\psi, \tau) = (1, 0)$ such that $\psi' < 1$, we have that,

(a) If $(1-\psi')(q_j - u_i) \geq \tau'$ and $\psi' < 1$, then

$$\pi_{s_j,b_i}(\psi',\tau') - \pi_{s_j,b_i}(\psi,\tau) = \tau' + \psi'\left[(1-\alpha)(q_j-u_i) + \alpha\frac{\tau'}{1-\psi'}\right] - (q_j - u_i)$$
$$\leq -\psi'(q_j-u_i) + \psi'\left[(1-\alpha)(q_j-u_i) + \alpha\frac{(1-\psi')(q_j-u_i)}{1-\psi'}\right] = 0.$$

(b) If $(1-\psi')(q_j - u_i) < \tau'$ or $\psi' = 1$, then

$$\pi_{s_j,b_i}(\psi',\tau') - \pi_{s_j,b_i}(\psi,\tau) = 0 - (q_j - u_i) < 0.$$

So we conclude that, for any $(\psi',\tau') \neq (\psi,\tau) = (1,0)$ and any $\varepsilon \geq 0$ we have that $\pi_{s_j,b_i}(\psi',\tau') - \pi_{s_j,b_i}(\psi,\tau) < \varepsilon$, which implies that $\pi_{s_j,b_i}(\cdot,\cdot)$ is u.s.c. in $(\psi,\tau) = (1,0)$.

Now clearly, if τ is sufficiently large, no transactions are made, which results in zero profits to the platform. So, without loss of generality, we can assume that the platform will only consider choosing $\tau \in [0,\overline{\tau}]$ for some $\overline{\tau} > 0$. In this case, for a given μ, the feasible set from the platform's problem is given by $[0,1]\times[0,\overline{\tau}]$, which is compact. Because any u.s.c. function defined on a compact set has a maximum (Leininger (1984)), we have that for a given μ there exists a (ψ,τ) that maximizes the platform's profits. Because the set of possible matches is finite, this implies that there exists a tuple (μ,ψ,τ) that maximizes the profits of the platform. ∎

Proof of lemma 2:
Let $\pi_\mu(\psi,\tau)$ be the profits from the platform when it implements matching μ and commissions (ψ,τ). Also, let $S_\mu(\psi,\tau) = \{s_j \in S; \mu(s_j) = b_i \in B \wedge (1-\psi)(q_{ij}-u_i) \geq c_j + \tau\}$, and let $\mu^*_{\psi,\tau}$ be the matching obtained when implementing algorithm 1 and the fees are given by (ψ,τ). Then, from theorem 1, we have that,

$$\pi_{\mu^*_{\psi,\tau}}(\psi,\tau) \geq \pi_\mu(\psi,\tau),$$

and

$$|S_{\mu^*_{\psi,\tau}}(\psi,\tau)| \geq |S_\mu(\psi,\tau)|,$$

for any matching μ.

Now, for any $\tau' \geq \tau$ such that $\tau' \leq \min_{(s_j,b_i)\in M_{\mu^*_{\psi,\tau}}}[(1-\psi)(q_{ij}-u_i) - c_j]$, define $\varepsilon \equiv \tau' - \tau \geq 0$. Then, because $S_\mu(\psi,\tau') \subseteq S_\mu(\psi,\tau)$ (i.e., *ceteris paribus*, the higher τ is, the lower the number of agents who transact) and because $S_{\mu^*_{\psi,\tau}}(\psi,\tau) = S_{\mu^*_{\psi,\tau}}(\psi,\tau')$, we have that

$$\begin{aligned}
\pi_{\mu^*_{\psi,\tau}}(\psi,\tau') &= \varepsilon\left(\frac{1-\psi(1-\alpha)}{1-\psi}\right)\underbrace{|S_{\mu^*_{\psi,\tau}}(\psi,\tau)|}_{\geq |S_\mu(\psi,\tau)|} + \underbrace{\pi_{\mu^*_{\psi,\tau}}(\psi,\tau)}_{\geq \pi_\mu(\psi,\tau)} \\
&\geq \varepsilon\left(\frac{1-\psi(1-\alpha)}{1-\psi}\right)|S_\mu(\psi,\tau)| + \pi_\mu(\psi,\tau) \\
&\geq \varepsilon\left(\frac{1-\psi(1-\alpha)}{1-\psi}\right)|S_\mu(\psi,\tau')| + \pi_\mu(\psi,\tau) \\
&= \varepsilon\left(\frac{1-\psi(1-\alpha)}{1-\psi}\right)|S_\mu(\psi,\tau')| + \sum_{(s_j,b_i)\in M_\mu(\psi,\tau)}\left[\psi\left[(1-\alpha)(q_{ij}-u_i)+\alpha\frac{c_j+\tau}{1-\psi}\right]+\tau\right] \\
&\geq \varepsilon\left(\frac{1-\psi(1-\alpha)}{1-\psi}\right)|S_\mu(\psi,\tau')| + \sum_{(s_j,b_i)\in M_\mu(\psi,\tau')}\left[\psi\left[(1-\alpha)(q_{ij}-u_i)+\alpha\frac{c_j+\tau}{1-\psi}\right]+\tau\right] \\
&= \pi_\mu(\psi,\tau'),
\end{aligned}$$

as we wanted to show. ■

Proof of theorem 2:

I) We first prove that *CPE* is always stable. Suppose that μ is *CPE*. Let

$$k \equiv \max\{i \in \{1,2,\cdots,\min\{n,m\}\}; (1-\psi)(q_i-u_i)-c_i \geq \tau\},$$

i.e., k is the maximum index such that buyers and sellers with that index are willing to transact with one another.

For each $i \leq k$, define

$$p_i = q_i - u_k,$$

i.e., p_i is the maximum that buyer k is willing to pay for product i. Then, for all $i \leq k$, define

$$\begin{aligned}
w_{s_i} &= (1-\psi)p_i - \tau \\
&= (1-\psi)(q_i-u_k) - \tau.
\end{aligned}$$

and

$$\begin{aligned}
w_{b_i} &= q_j - p_j, \quad \text{for some } j \leq k \\
&= u_k,
\end{aligned}$$

For $i \in \{k+1,\cdots,n\}$, define

$$w_{b_i} = u_i,$$

and $i \in \{k+1,\cdots,m\}$, define

$$w_{s_i} = c_i.$$

We now show that $\{w_{b_i}\}_{b_i\in B}$ and $\{w_{s_j}\}_{s_j\in S}$ satisfy conditions i), ii) and iii) from definition 3:

i) (Individual rationality) For all $i \leq k$,

$$w_{b_i} = u_k \geq u_i,$$

and

$$w_{s_i} = (1-\psi)(q_i - u_k) - \tau \geq (1-\psi)(q_i - u_i) - \tau \geq c_i.$$

For $i \in \{k+1, \cdots, n\}$, we have that

$$w_{b_i} = u_i$$

and for $i \in \{k+1, \cdots, m\}$, we have that

$$w_{s_i} = c_i,$$

so individual rationality is trivially satisfied in these cases.

ii) (Feasibility) We only have $w_{s_i} > c_i$ or $w_{b_i} > u_i$ for $i \leq k$. In these cases feasibility is satisfied by construction.

iii) (No Blocking Coalition) Now, suppose by way of contradiction that there is a $s_{j'} \in S$ and $b_{i'} \in B$, and $(\tilde{w}_{s_{j'}}, \tilde{w}_{b_i}) \in \mathbb{R}^2$ such that

$$q_{j'} - u_{i'} - r > \frac{c_{j'} + \tau}{1-\psi},$$

and that there is a $\tilde{p} \in [\frac{c_{j'}+\tau}{1-\psi}, q_{j'} - u_{i'} - r]$ such that

$$\tilde{w}_{s_{j'}} \equiv (1-\psi)\tilde{p} - \tau > w_{s_{j'}}$$

and

$$\tilde{w}_{b_{i'}} \equiv q_{j'} - \tilde{p} - r > w_{b_{i'}}.$$

Then there are four possible cases to consider:

Case 1) Suppose that $i' \leq k$ and $j' \leq k$. Then $w_{b_{i'}} = u_k$ and $w_{s_{j'}} = (1-\psi)(q_{j'} - u_k) - \tau$. Therefore, $\tilde{w}_{s_{j'}} > w_{s_{j'}}$ and $\tilde{w}_{b_{i'}} > w_{b_{i'}}$ imply that

$$\begin{aligned} &(1-\psi)\tilde{p} - \tau > (1-\psi)(q_{j'} - u_k) - \tau \\ \iff &\tilde{p} > q_{j'} - u_k \end{aligned} \tag{15}$$

and

$$\begin{aligned} &q_{j'} - \tilde{p} - r > u_k \\ \iff &\tilde{p} < q_{j'} - r - u_k. \end{aligned} \tag{16}$$

But together, inequalities (15) and (16) imply that

$$q_{j'} < q_{j'} - r,$$

a contradiction with $r \geq 0$.→←

Cbse 2) Suppose that $i' > k$ and $j' \leq k$. Then $w_{b_{i'}} = u_{i'} \geq u_k$, and $w_{s_{j'}} = (1-\psi)(q_{j'} - u_k) - \tau$. Therefore, $\tilde{w}_{s_{j'}} > w_{s_{j'}}$ and $\tilde{w}_{b_{i'}} > w_{b_{i'}}$ imply that

$$\begin{aligned} &(1-\psi)\tilde{p} - \tau > (1-\psi)(q_{j'} - u_k) - \tau \\ \iff &\tilde{p} > q_{j'} - u_k \end{aligned} \tag{17}$$

and

$$\begin{aligned} &q_{j'} - \tilde{p} - r > u_{i'} > u_k \\ \Rightarrow &\tilde{p} < q_{j'} - r - u_k. \end{aligned} \tag{18}$$

But together, inequalities (17) and (18) imply that

$$q_{j'} < q_{j'} - r,$$

a contradiction with $r \geq 0$.$\rightarrow\leftarrow$

Ccse 3) Suppose that $i' \leq k$ and $j' > k$. Then $w_{b_{i'}} = u_k$, and $w_{s_{j'}} = q_{j'}$. Therefore, $\tilde{w}_{s_{j'}} > w_{s_{j'}}$ and $\tilde{w}_{b_{i'}} > w_{b_{i'}}$ imply that

$$\begin{aligned} &(1-\psi)\tilde{p} - \tau > q_{j'} \\ \iff &\tilde{p} > \frac{q_{j'} + \tau}{1-\psi} \end{aligned} \tag{19}$$

and

$$\begin{aligned} &q_{j'} - \tilde{p} - r > u_k \\ \Rightarrow &\tilde{p} < q_{j'} - r - u_k. \end{aligned} \tag{20}$$

But together, inequalities (19) and (20) imply that

$$\begin{aligned} &(1-\psi)(q_{j'} - u_{j'} - r) > c_{j'} + \tau \\ \Rightarrow &(1-\psi)(q_{j'} - u_{j'}) > c_{j'} + \tau, \end{aligned}$$

a contradiction with $j' > k$.$\rightarrow\leftarrow$

Cdse 4) Suppose $i' > k$ and $j' > k$. Then $w_{b_{i'}} = u_{i'} \geq u_k$, and $w_{s_{j'}} = c_{j'}$. Therefore, $\tilde{w}_{s_{j'}} > w_{s_{j'}}$ and $\tilde{w}_{b_{i'}} > w_{b_{i'}}$ imply that

$$\begin{aligned} &(1-\psi)\tilde{p} - \tau > c_{j'} \\ \iff &\tilde{p} > \frac{c_{j'} + \tau}{1-\psi} \end{aligned} \tag{21}$$

and

$$\begin{aligned} &q_{j'} - \tilde{p} - r > u_{i'} \\ \Rightarrow &\tilde{p} < q_{j'} - r - u_{i'}. \end{aligned} \tag{22}$$

Together, inequalities (21) and (22) imply that

$$\begin{aligned}&(1-\psi)(q_{j'}-u_{i'}-r) > c_{j'}+\tau\\ \Rightarrow&(1-\psi)(q_{j'}-u_{i'}) > c_{j'}+\tau.\end{aligned} \tag{23}$$

Defining $l=\min\{j',i'\}$, we have that inequalities (23) implies that

$$(1-\psi)(q_l-u_l) > c_l+\tau,$$

a contradiction with $l \geq k$.$\rightarrow\leftarrow$

II) Now let us show that, when $r=0$, a matching μ is stable iff it is *CPE*. We have already seen that, if a matching is *CPE* it is stable for any $r \geq 0$, including $r=0$. So it only remains to show that, if $r=0$ and μ is not *CPE*, then it is *not* stable.

So, suppose by way of contradiction that μ is not *CPE* and suppose by way of contradiction that it is stable. Then, letting

$$k \equiv \max\{i \in \{1,2,\cdots,\min\{n,m\}\}; (1-\psi)(q_i-u_i)-c_i \geq \tau\},$$

there must be a s_j with $j \leq k$ such that $\mu(s_j)=b_i$ with $i>k$ or $\mu(s_j)=s_j$. In either case, feasibility implies that

$$w_{s_j} < (1-\psi)(q_j-u_k)-\tau.$$

But $\mu(s_j)=b_i$ with $i>k$ implies that there is an $i \leq k$ such that $\mu(b_i)=s_{j'}$ with $j'>k$ or $\mu(b_i)=b_i$. In either case, feasibility implies that

$$w_{b_i} < u_k.$$

In this case, if seller s_j was matched with buyer b_i and charged a price $p_j=q_j-u_k$, both seller s_j and buyer b_i would be made better off, a contradiction with μ being *CPE*.

III) Now, let us prove that if $(1-\psi)r \geq \max_j\{(1-\psi)q_j-c_j\}-(1-\psi)\min_i\{u_i\}-\tau$, any matching μ is stable.

Let μ be an arbitrary matching. For each seller $s_j \in S$ and each buyer $b_i \in B$ define $w_{s_j} \in \mathbb{R}$ and $w_{b_i} \in \mathbb{R}$, respectively, such that:

i) If $\mu(s_j)=b_i \in B$ and $q_j-u_i \geq \frac{c_j+\tau}{1-\psi}$,

$$w_{s_j}=(1-\psi)p_j-\tau,$$

$$w_{b_i}=q_j-p_j,$$

where

$$p_j=\frac{q_j-u_i-\frac{c_j+\tau}{1-\psi}}{2}+\frac{c_j+\tau}{1-\psi}.$$

ii) If $\mu(s_j)=s_j$ or $\mu(s_j)=b_i \in B$ and $q_j-u_i < \frac{c_j+\tau}{1-\psi}$, then $w_{s_j}=c_j$.

iii) If $\mu(b_i)=b_i$ or $\mu(b_i)=s_j \in S$ and $q_j-u_i < \frac{c_j+\tau}{1-\psi}$, then $w_{b_i}=u_i$.

Clearly, $(w_{s_j})_{s_j \in S}$ and $(w_{b_i})_{b_i \in B}$ satisfy the properties of individual rationality and feasibility. Now, suppose by way of contradiction that there is a $(s_{j'}, b_{i'}) \in S \times B$ a $(\tilde{w}_{s_j}, \tilde{w}_{b_i}, \tilde{p}) \in \mathbb{R}^3$ such that

$$\tilde{w}_{s_j} = (1-\psi)\tilde{p} - \tau > w_{s_j} \geq c_j \tag{24}$$

and

$$\tilde{w}_{b_i} = q_j - \tilde{p} - r > w_{b_i} \geq u_i. \tag{25}$$

Then, inequalities (24) and (25) imply that

$$\frac{c_j + \tau}{1-\psi} < q_j - r - u_i$$
$$\iff (1-\psi)r < (1-\psi)q_j - c_j - (1-\psi)u_i - \tau,$$

a contradiction with the hypothesis that $(1-\psi)r \geq \max_j \{(1-\psi)q_j - c_j\} - (1-\psi)\min_i\{u_i\} - \tau$.$\rightarrow\leftarrow$

■

B *PAM* and algorithm 1 when sellers are ordered in ascending order of their costs and products are differentiated

Suppose that

$$c_1 \leq c_2 \leq \cdots \leq c_m,$$

and define *PAM* as matching buyer b_1 with seller s_1, buyer b_2 with seller s_2, and so on. The following example illustrates how this version of *PAM* is not necessarily *CPE* when products are differentiated, and moreover, how the matching obtained through algorithm 1 is not necessarily profit-maximizing.

Example 3. *(When products are differentiated PAM is not necessarily CPE, and algorithm 1 is not necessarily profit-maximizing) Consider a market where the set of sellers is given by $\{s_1, s_2, s_3\}$, and the set of buyers is given by $\{b_1, b_2, b_3\}$. Suppose that products are only vertically differentiated, so that $q_{ij} = q_j \in \mathbb{R}_+$ for all $i \in \{1,2,3\}$. Sellers' quality and costs are given by*

$$(q_1, q_2, q_3) = (4, 6, 14) \quad \textit{and} \quad (c_1, c_2, c_3) = (1, 3, 6),$$

respectively. Customers' outside options are given by

$$(u_1, u_2, u_3, u_4) = (0, 2, 5).$$

If PAM is implemented, then

$$s_1 : b_1 \quad s_2 : b_2, \quad s_3 : b_3.$$

If the platform charges commission fees $\psi = 1/2$ and $\tau = 0$, then this matching would yield only one transaction (the one between seller s_1 and buyer b_1) generating a total profit of \$1.5 to the platform, and a total surplus of

$$q_1 - c_1 - u_1 = 3.$$

But if it implemented the matching from algorithm 1:

$$s_3 : b_1, \quad s_1 : b_2, \tag{26}$$

then 2 transactions would take place: the one between seller s_3 and buyer b_1 and the one between seller s_1 and buyer b_2. This would generate a total revenue of \$7.5 to the platform and a total surplus of

$$(q_3 - c_3 - u_1) + (q_1 - c_1 - u_2) = 9,$$

from which we conclude that PAM is not necessarily CPE.

Now notice that, if the platform implemented the following matching instead

$$s_2 : b_1, \quad s_3 : b_2,$$

it would get a total profit of \$9, which is greater than the profit of \$7.5 obtained by implementing algorithm 1. Therefore, when products are differentiated, algorithm 1 is not necessarily profit-maximizing.

C The Hungarian Method

In this section we briefly describe the Hungarian method. More details about the algorithm can be found in Schrijver (2004) or in https://brilliant.org/wiki/hungarian-matching/.

Suppose we are interested in finding the matching that maximizes the platform's revenues. Then we write a $m \times m$ matrix in which each entry i, j gives the marginal revenue obtained if buyer b_i was matched with seller s_j, if $i \leq n$. Notice that, if $(1-\psi)(q_{ij} - c_j) < u_i + \tau$, the i, j entry from this matrix would be zero, as these agents would not transact if matched with one another. But if $(1-\psi)(q_{ij} - c_j) \geq u_i + \tau$, then, in the event these agentes were matched to one another, they would transact, so the i, j entry from the matrix corresponding to this marginal revenue would be equal to

$$\psi \left[(1-\alpha)(q_{ij} - u_i) + \alpha \frac{c_j + \tau}{1-\psi} \right] + \tau.$$

For $i > n$, set the i, j entry equal to zero (these zeros would represent the profits generated to the platform from a seller who ended up unassigned).

If we were interested in finding a *CPE* matching, we would fill the i, j entry, instead, by the total surplus generated from this transaction, which would be given by $q_{ij} - c_j - u_i$. Notice that this total surplus includes the fraction that is captured by the platform through its commissions.

Algorithm 2. *(The Hungarian method)*

1. *Find the highest entry from each row, and subtract it from each entry within the row. This will make the highest entry in the row now equal to 0.*

2. *Now find the highest element from each column and subtract it from each element within the column.*

3. *Draw lines through the rows and columns that have the 0 entries such that the fewest lines possible are drawn.*

4. *If there are n lines drawn, an optimal assignment of zeros is possible and the algorithm is finished. If the number of lines is less than n, then the optimal number of zeroes is not yet reached. Go to the next step.*

5. *Find the highest entry not covered by any line. Subtract it from each row that isn't crossed out, and then add it to each column that is crossed out. Then, go back to Step 3.*

6. *The final assignment will be where the 0's are in the matrix such that only one 0 per row and column is part of the assignment.*

It can be shown that this algorithm is $\mathcal{O}(m^4)$, and that modifications to the algorithm can further improve its convergence rate (Schrijver (2004)).

D Optimal commissions and optimal matching

In the previous sections we assumed the commission fees ψ and τ to be exogenous. But in reality one would expect commissions to be set by the platform along with the matching function μ. In this case, the platform would choose ψ, τ and μ to maximize

$$\max_{\mu,\psi,\tau} \tau|S_{\mu,\psi,\tau}| + \sum_{s_j \in S_{\mu,\psi,\tau}} \psi p_j^*(\mu(s_j), \psi, \tau), \tag{27}$$

where

$$S_{\mu,\psi,\tau} \equiv \{s_j \in S; \mu(s_j) = b_i \in B \wedge (1-\psi)(q_{ij} - u_i) \geq c_j + \tau\}$$

corresponds to the set of sellers who end up transacting given the fees (ψ, τ) and matching function μ, and $p_j^*(\mu(s_j), \psi, \tau)$ is the equilibrium price given by equation 3.

It can be shown that a maximum to this problem always exists. Indeed, for a given matching μ, we can show that the platform's objective function is upper semicontinuous in $(\psi, \tau) \in [0,1] \times \mathbb{R}_+$. Moreover, we can limit τ to be less than or equal to a constant $\overline{\tau}$, since, for τ sufficiently high, no transactions are made, so the platform earns zero profits. Because, for a given μ, the objective function is upper semicontinuous, and because we only consider (ψ, τ) defined on the compact set $[0,1] \times [0, \overline{\tau}]$, it then follows that the problem has a maximum for a given μ, since every upper semicontinuous function defined on a compact set has a maximum (Leininger (1984)). Because the set of possible matchings is finite, we conclude that a maximum to problem 27 exists.

Proposition 6. *There exists a pair of commission rates (ψ, τ) and a matching function μ that maximizes the profits of the platform.*

Proof: In the Appendix. ∎

The next lemma states that, under an optimal matching, at least one agent must be exactly indifferent between transacting and not transacting, otherwise, the platform would be able to increase its fees without reducing the number of transactions. In order to state the lemma more formally, let $\mu^*_{\psi,\tau}$ be a matching that maximizes revenues for a given (ψ, τ). For a given matching μ, let

$$M_\mu(\psi, \tau) \equiv \{(s_j, b_i) \in S \times B; \mu(s_j) = b_i \wedge (1-\psi)(q_{ij} - u_i) \geq c_j + \tau\},$$

i.e., $M_\mu(\psi,\tau)$ represents all the matches associated with μ, excluding those in which agents do not transact. When (ψ,τ) are known and exogenous, we sometimes write M_μ instead of $M_\mu(\psi,\tau)$ to simplify the notation.

Lemma 1. *If $(\mu^*_{\psi,\tau},\psi,\tau)$ is a solution to the platform's objective function 27, we must have $(1-\psi)(q_{ij}-u_i) = c_j+\tau$ for at least one $(s_j,b_i)\in M_{\mu^*_{\psi,\tau}}(\psi,\tau)$.*

Proof: If all agents who transact were strictly better off transacting as opposed to not transacting, the platform would be able to marginally increase ψ or τ, thus increasing its revenues per transactions, without reducing the number of transactions. ∎

The next lemma states that, if products have homogenous quality and, for a pair of commission rates (ψ,τ), a matching μ^* maximizes profits, and, moreover, all transactions from this matching generate a positive surplus to buyers and sellers, then, if we were to marginally increment τ, the matching μ^* would still be profit-maximizing.

Lemma 2. *Suppose that $q_{ij}=\overline{q}$ for all $i\leq n$ and all $j\leq m$. Let $\mu^*_{\psi,\tau}$ be the profit-maximizing matching obtained through algorithm 1 when commissions are given by (ψ,τ). Suppose, in addition, that $\tau<(1-\psi)(\overline{q}-u_i)-c_j$ for all $(s_j,b_i)\in M_{\mu^*_{\psi,\tau}}(\psi,\tau)$. Then, for any $\tau'\geq\tau$ such that $\tau'\leq\min_{(s_j,b_i)\in M_{\mu^*_{\psi,\tau}}(\psi,\tau)}[(1-\psi)(q_{ij}-u_i)-c_j]$, we have that $\mu^*_{\psi,\tau'}=\mu^*_{\psi,\tau}$.*

Proof: In the Appendix. ∎

If we assume that ψ is set exogenously and that $q_{ij}=\overline{q}$ for all $i\leq n$ and all $j\leq m$, we can use theorem 1 and lemmas 1 and 2 to build an algorithm to find the optimal $\tau\geq 0$ and optimal matching $\mu^*_{\psi,\tau}$. The algorithm basically finds all the points in $\tau\geq 0$ such that, after implementing the matching $\mu^*_{\psi,\tau}$, at least one of the sellers who transacts under this matching is indifferent between transacting and not transacting, as those points will be our only candidates for an optimum.

Algorithm 3. *Let $\mu^*_{\psi,\tau}$ be the matching algorithm that maximizes the platform's profits when the fees are given by (ψ,τ). For a given $\psi\in[0,1)$, perform the following algorithm.*

i) Initialize $k=0$ and $\tau=0$.

ii) Compute $\mu^*_{\psi,\tau}$. If $M_{\mu^*_{\psi,\tau}}\neq\emptyset$, proceed to the next step, else, stop the algorithm.

iii) Set $k=k+1$. Define

$$\tau_k\equiv\min_{(s_j,b_i)\in M_{\mu^*_{\psi,\tau}}(\psi,\tau)}(1-\psi)(q_{ij}-u_i)-c_j$$

If $\tau_k>\tau$, redefine $\tau=\tau_k$, else (if $\tau_k=\tau$), redefine $\tau=\tau_k+\varepsilon$, where ε is arbitrarily small.[13] Then, repeat step ii).

If $k=0$, then any $\psi\in[0,1]$ and any matching μ generates zero profits to the platform. If $k>0$, choose the commission

$$\tau^*=\underset{\tau\in\{\tau_1,\tau_2,\cdots,\tau_k\}}{\arg\max}\;\pi(\mu^*_{\psi,\tau},\psi,\tau)$$

[13] More precisely, define $\tilde{M}(\psi,\tau)\equiv\{(s_j,b_i)\in M_{\mu^*_{\psi,\tau}}(\psi,\tau);(1-\psi)(q_{ij}-u_i)-c_j>\tau_k\}$, i.e., $\tilde{M}(\psi,\tau)$ corresponds to the matched pairs of buyers and sellers that would still transact under $\mu^*_{\psi,\tau}$ if τ was marginally increased. Then, if $\tilde{M}(\psi,\tau)\neq\emptyset$, set an ε such that

$$0<\varepsilon<\min_{(s_j,b_i)\in\tilde{M}(\psi,\tau)}(1-\psi)(q_{ij}-u_i)-c_j-\tau_k),$$

else (if $\tilde{M}(\psi,\tau)=\emptyset$), set $\varepsilon=1$.

and the matching μ^*_{ψ,τ^*}.

Theorem 3. *Suppose that* $q_{ij} = \overline{q}$ *for all* $i \leq n$ *and all* $j \leq m$. *For a given* $\psi \in [0,1)$, *let* τ^* *and* μ^*_{ψ,τ^*} *be the flat fee and matching function, respectively, obtained after implementing algorithm 3. Define* $S_\mu(\psi,\tau) = \{s_j \in S; \mu(s_j) = b_i \in B \wedge (1-\psi)(\overline{q} - u_i) \geq c_j + \tau\}$. *Then,*

$$(\mu^*_{\psi,\tau^*}, \tau^*) = \arg\max_{\mu,\tau} \tau|S_\mu(\psi,\tau)| + \sum_{s_j \in S_\mu(\psi,\tau)} \psi p^*_j(\mu(s_j), \psi, \tau).$$

Proof: The proof follows immediately from theorem 1 and lemmas 1 and 2. ∎

Though algorithm 3 does not simultaneously find the optimal ψ and optimal τ, one could create a grid for $\psi \in [0,1)$, and then perform algorithm 3 for each ψ in the grid, and then select the ψ from the grid that generates the maximum revenue for the platform.

Notice that algorithm 3 does not necessarily yield the optimum policy for the platform if products are differentiated, as theorem 3 only applies to the case in which $q_{ij} = \overline{q}$ for all $i \leq n$ and all $j \leq m$. As an alternative, one could simply create a grid for both ψ and τ, and find the optimum match for each combination of (ψ,τ) within the grid. The advantage of using algorithm 3, however, is that it makes it more likely that a necessary condition for optimization is met: that at least one pair of buyer and seller is exactly indifferent between transacting and not transacting. Moreover, through this method one does not need to specify an upper bound for the values that τ can assume (notice that τ can be any positive number).

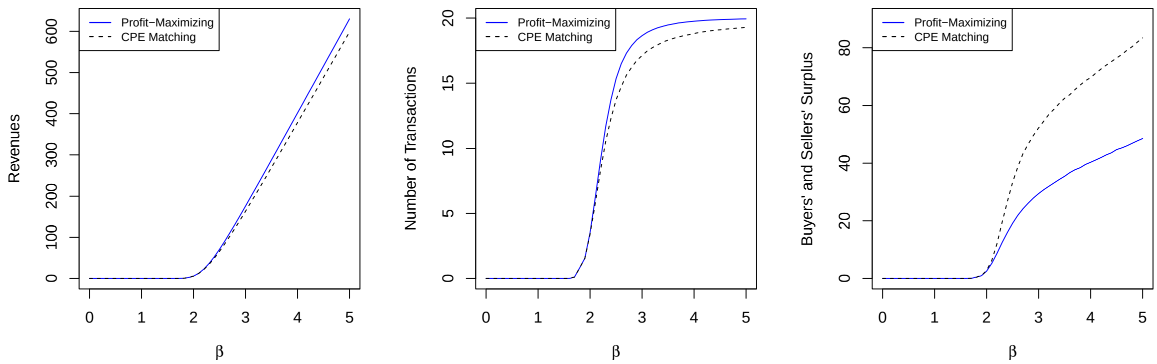
With these results we can use the Hungarian method to simulate the platform's optimal matching and fees (ψ,τ). We can then compare the platform's revenues and number of transactions when implementing the optimal policy, vs. the case in which it seeks to maximize revenues, subject to the constraint that the matching must be *CPE*.

Similar to the previous section, we assume that $c_j \sim N(10,1)$ and $u_i \sim N(15,1)$ for all $j \leq m$ and all $i \leq n$. Moreover,

$$q_{ij} = \xi_{ij} + \beta c_j,$$

where $\xi_{ij} \sim N(0,1)$ and $\beta \geq 0$. The parameter β captures the extend to which quality is correlated with production costs.

Figure 6 shows the differences in revenues, number of transactions and sum of buyers' and sellers' surplus obtained when the platform attempts to implement its optimum policy by following algorithm 3 vs. the case in which it attempts to implement profit-maximizing commissions subject to the constraint that the matching must be *CPE*, by also following algorithm 3 but with the difference that, at each step of the algorithm, it implements a matching that maximizes the sum of customers' and sellers' surplus, not the platform's profits. The simulations suggest that those differences can be quite significant and they do not disappear for arbitrarily high values of β. This happens because, as β increases, the platform also increases its fees, until it reaches a point in which "demand crosses supply", which, from corollary 3, would imply that the platform has strict incentives to implement a profit-maximizing matching, as opposed to a *CPE* matching. Though corollary 3 only applies to the case in which products have homogeneous quality, our simulations indicate that the intuition of this result seems to extend to cases in which products are differentiated.



(a) Difference in revenues between a proft-maximizing and a *CPE* matching

(b) Difference in number of transactions between a proft-maximizing and a *CPE* matching

(c) Difference in Buyer and Seller surplus between a proft-maximizing and a *CPE* matching

Figure 6: Plots depicting simulated differences between a profit-maximizing and a *CPE* matching for different levels of β, assuming that $n = m = 20$ (so there can be no more than 20 transactions in total). 10,000 simulations were used to compute the averages from each β in the grid.

E Stability when agents can transact outside the platform

In this section we analyze the performance of Positive Assortative Matching (*PAM*) in terms of *stability* when $\psi = 0$ and $\tau \geq 0$. The notion of stability presented in this section is similar to the one studied in section 5, with the difference that now, when agents pay the search cost to form a blocking coalition, they transact outside the platform, say, because the search results given by the platform are too limited, so that agents can only seek new matches using tools not provided by the platform. Letting θ denote the search cost, we show that, provided that $\psi = 0$, *PAM* is the matching most likely to achieve *stability* in the sense that it is less likely to have a buyer or a seller willing to pay a search cost θ to form a *blocking coalition*.

Suppose that the platform charges commission fees (ψ, τ) and implements the matching μ. Suppose that, after the platform implements this matching, each buyer $b_i \in B$ can pay a search cost θ to be matched with any of the sellers in the market (including seller $\mu(b_i)$). The cost θ could also capture potential contractual costs associated with making transactions outside the platform (see the practical examples given in section 5). If, during this second round, buyer b_i chooses to pay θ to be matched with seller s_j, seller s_j will then choose whether to transact with its current match $\mu(s_j)$ and pay the commission fees owed to the platform, or transact with buyer b_i and pay no fees to the platform.

We say a matching μ is *stable* if no buyer has incentives to incur in the search cost θ to bypass the intermediation of the platform.

Definition 5. ***(Stability)*** *Given $\psi = 0$ and $\tau \geq 0$, and given the search cost θ, a matching μ is stable if, for every $s_j \in S$ there is a payoff $w_{s_j} \in \mathbb{R}$ and for every $b_i \in B$ there is a payoff $w_{b_i} \in \mathbb{R}$ such that:*

i) (Individual rationality) $w_{s_j} \geq c_j$ $\forall s_j \in S$ and $w_{b_i} \geq u_i$ $\forall b_i \in B$ (i.e., agents' payoffs are greater than or equal to the their outside option).

ii) (Feasibility) If $\mu(s_j) = b_l \in B$, then $w_{b_l} + w_{s_j} \leq q_j - \tau$ if $w_{s_j} > c_j$ or $w_{b_l} > u_l$. If $\mu(s_j) = s_j$, then $w_{s_j} = c_j$, and if $\mu(b_i) = b_i$, then $w_{b_i} = u_i$ (i.e., unmatched agents get their outside option).

iii) (No Blocking Coalition) There is no $(s_j, b_i) \in S \times B$ and no $(\tilde{w}_{s_j}, \tilde{w}_{b_i}) \in \mathbb{R}^2$ such that:

1) $\tilde{w}_{s_j} + \tilde{w}_{b_i} \leq q_j - \theta$

2) $\tilde{w}_{s_j} > w_{s_j}$ *and* $\tilde{w}_{b_i} > w_{b_i}$.

Clearly, if the search cost θ is sufficiently high, every matching is stable. In this case, for a given $\tau \geq 0$, the platform does not need to worry about implementing a matching that incentivizes agents to follow through its recommendations.

Theorem 4. *Suppose that the platform chooses $\psi = 0$ and $\tau \geq 0$, and suppose that $q_1 - c_1 - u_1 \geq \tau$.*[14]

I) A necessary and sufficient condition for PAM to be stable is that $\theta \geq \tau$. Moreover, if a matching μ is stable, then so is PAM.

II) If

$$\theta \geq \max_j\{q_j - c_j\} - \min_i\{u_i\}. \tag{28}$$

then any matching μ is stable.

Proof:

I) To prove the first part of the theorem, we will show that a necessary condition for **any** matching μ to be stable (including PAM) is that $\theta \geq \tau$, and then show that a sufficient condition for PAM to be stable is that $\theta \geq \tau$. This implies that, if a an arbitrary matching μ is stable, then $\theta \geq \tau$, so that PAM is also stable. Throughout the proof, we will assume $\psi = 0$.

Necessity: Let μ be an arbitrary matching. For each seller $s_j \in S$ we define a $w_{s_j} \in \mathbb{R}$ and for every buyer $b_i \in B$ we define a $w_{b_i} \in \mathbb{R}$ such that:

i) *(Individual rationality):* $w_{s_j} \geq c_j \; \forall s_j \in S$ and $w_{b_i} \geq u_i \; \forall b_i \in B$.

ii) *(Feasibility):* If $\mu(s_j) = b_l \in B$, then $w_{b_l} + w_{s_j} \leq q_i - \tau$ if $w_{s_j} > c_j$ or $w_{b_l} > u_i$. If $\mu(s_j) = s_j$, then $w_{s_j} = c_j$, and if $\mu(b_i) = b_i$, then $w_{b_i} = u_i$.

Suppose that $\theta < \tau$. Then there exists a $\varepsilon > 0$ such that $\theta + \varepsilon < \tau$.

Let

$$FM_\mu(\tau) \equiv \{(s_j, b_i) \in S \times B; \mu(s_j) = b_i \text{ and } c_j + u_i \leq q_j - \tau\},$$

i.e., $FM_\mu(\tau)$ represents all the matches associated with μ, excluding those that do not generate transactions (i.e., FM_μ could be interpreted as the "*feasible matches*" associated with μ, when the platform chooses $\psi = 0$ and $\tau \geq 0$).

If $FM_\mu = \emptyset$ then μ is not stable. Indeed, if $FM_\mu = \emptyset$ we must have

$$w_{s_1} = c_1 \quad \text{and} \quad w_{b_1} = b_1.$$

So defining

$$\tilde{w}_{s_1} \equiv \frac{\varepsilon}{2} + w_{s_1}$$

and

$$\tilde{w}_{b_1} \equiv \frac{\varepsilon}{2} + w_{b_1},$$

[14] $q_1 - c_1 - u_1 \geq \tau$ ensures that the platform can intermediate at least one transaction (assuming that θ is not too low, so that everyone would rather transact outside of the platform).

we have, from the assumption $\tau \leq q_1 - c_1 - b_1$, that

$$\tilde{w}_{s_1} + \tilde{w}_{b_1} = \varepsilon + c_1 + u_1 \leq \varepsilon + q_1 - \tau < q_1 - \theta,$$

which implies that μ is not stable.

So suppose $FM_\mu(\tau) \neq \emptyset$ and let $(s_j, b_i) \in FM_\mu(\tau)$. Then, for any $(w_{s_j}, w_{b_i}) \in \mathbb{R}^2$ such that

$$\begin{aligned} w_{s_j} &\geq c_j \\ w_{b_i} &\geq u_i \\ w_{s_j} + w_{b_i} &\leq q_j - \tau, \end{aligned}$$

define

$$\begin{aligned} \tilde{w}_{s_j} &\equiv \frac{\varepsilon}{2} + w_{s_j} \\ \tilde{w}_{b_i} &\equiv \frac{\varepsilon}{2} + w_{b_i}. \end{aligned}$$

Then, we have that

$$\tilde{w}_{s_j} + \tilde{w}_{b_i} = \varepsilon + w_{s_j} + w_{b_i} \leq \varepsilon + q_j - \tau < q_j - \theta,$$

so that μ is not stable.

Sufficiency: Suppose that μ is *PAM*. Let

$$k \equiv \max\{i \in \{1, 2, \cdots, \min\{n, m\}\}; q_i - c_i - u_i \geq \tau\},$$

i.e., k is the maximum index obtained by matching sellers with buyers with the same index, such that matched agents have incentives to transact.

Then, for all $i \leq k$, define

$$\begin{aligned} w_{b_i} &\equiv u_k \\ w_{s_i} &\equiv q_i - \tau - u_k. \end{aligned}$$

For $i \in \{k+1, \cdots, n\}$, define

$$w_{b_i} = u_i,$$

and $i \in \{k+1, \cdots, m\}$, define

$$w_{w_i} = c_i.$$

We now show that $\{w_{b_i}\}_{b_i \in B}$ and $\{w_{s_j}\}_{s_j \in S}$ satisfy conditions i) and ii) from definition 5:

i) (Individual rationality) For all $i \leq k$,

$$w_{b_i} = u_k \geq u_i,$$

and

$$w_{s_i} = q_i - \tau - u_k \geq q_i - \tau - u_i \geq c_i.$$

For $i \in \{k+1, \cdots, n\}$, we have that

$$w_{b_i} = u_i$$

and for $i \in \{k+1, \cdots, m\}$, we have that

$$w_{s_i} = c_i,$$

so individual rationality and feasibility are trivially satisfied in these cases.

ii) (Feasibility) We only have $w_{s_i} > c_i$ or $w_{b_i} > u_i$ for $i \leq k$. In these cases, we have that

$$w_{b_i} + w_{s_i} = q_i - \tau - u_k + u_k = q_i - \tau,$$

so the feasibility condition is also satisfied.

Now suppose by contradiction that there is a $s_{j'} \in S$ and $b_{i'} \in B$, and $(\tilde{w}_{s_{j'}}, \tilde{w}_{b_i}) \in \mathbb{R}^2$ such that

1) $\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} \leq q_{i'} - \theta$
2) $\tilde{w}_{s_{j'}} > w_{s_{j'}}$ and $\tilde{w}_{b_{i'}} > w_{b_{i'}}$.

Then there are four possible cases to consider:

Case 1) Suppose that $i' \leq k$ and $j' \leq k$. Then $w_{b_{i'}} = u_i$ and $w_{s_{j'}} = q_{j'} - \tau - u_k$. Therefore, $\tilde{w}_{s_{j'}} > w_{s_{j'}}$ and $\tilde{w}_{b_{i'}} > w_{b_{i'}}$ implies that

$$\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} > w_{s_{j'}} + w_{b_{i'}} = q_{j'} - \tau > q_{j'} - \theta,$$

a contradiction with the feasibility condition $\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} \leq q_{i'} - \theta$.$\rightarrow\leftarrow$

Cbse 2) Suppose $i' > k$ and $j' \leq k$. Then $w_{b_{i'}} = u_{i'} \geq u_k$, and $w_{s_{j'}} = q_{j'} - \tau - u_k$, so that

$$\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} > w_{s_{j'}} + w_{b_{i'}} = q_{j'} - \tau - u_k + u_{i'} > q_{j'} - \tau > q_{j'} - \theta,$$

a contradiction with the feasibility condition $\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} \leq q_{i'} - \theta$.$\rightarrow\leftarrow$

Ccse 3) Suppose $i' \leq k$ and $j' > k$. Then $w_{b_{i'}} = u_k \geq u_{i'}$, and $w_{s_{j'}} = c_{j'}$, so that

$$\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} > w_{s_{j'}} + w_{b_{i'}} = c_{j'} + u_k \geq c_{j'} + u_{i'} > q_{j'} - u_{j'} - \tau + u_{i'} > q_{j'} - \theta,$$

where the last inequality comes form the fact that $u_{j'} \geq u_{i'}$, since $j' > i'$, and from the fact that $\theta > \tau$. So again, we obtain a contradiction with the feasibility condition $\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} \leq q_{i'} - \theta$.$\rightarrow\leftarrow$

Cdse 4) Suppose $i' > k$ and $j' > k$. Then $w_{b_{i'}} = u_{i'} \geq u_k$, and $w_{s_{j'}} = c_{j'}$. Moreover, because $j' \geq k+1$, we have that $q_{k+1} - c_{k+1} \geq q_{j'} - c_{j'}$. Therefore,

$$\tau > q_{k+1} - c_{k+1} - u_{k+1} \geq q_{j'} - c_{j'} - u_{i'} = q_{j'} - w_{s_{j'}} - w_{b_{i'}} > q_{j'} - \tilde{w}_{s_{j'}} - \tilde{w}_{b_{i'}}$$
$$\Rightarrow \tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} > q_{j'} - \tau > q_{j'} - \theta.$$

a contradiction with the feasibility condition $\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} \leq q_{i'} - \theta$.$\rightarrow\leftarrow$

II) Now let us prove that if $\theta \geq \max_j\{q_j - c_j\} - \min_i\{u_i\}$, any matching μ is stable.

Let μ be an arbitrary matching. For each seller $s_j \in S$ we define a $w_{s_j} \in \mathbb{R}$ and for every buyer $b_i \in B$ we define a $w_{b_i} \in \mathbb{R}$ such that:

i) *(Individual rationality):* $w_{s_j} \geq c_j \ \forall s_j \in S$ and $w_{b_i} \geq u_i \ \forall b_i \in B$. If $\mu(s_j) = s_j$, then $w_{s_j} = c_j$, and if $\mu(b_i) = b_i$, then $w_{b_i} = u_i$.

ii) *(Feasibility):* If $\mu(s_j) = b_l \in B$, then $w_{b_l} + w_{s_j} \leq q_i - \tau$ if $w_{s_j} > c_j$ or $w_{b_l} > u_i$.

Now suppose that there is a $s_{j'} \in S$ and $b_{i'} \in B$, and $(\tilde{w}_{s_{j'}}, \tilde{w}_{b_i}) \in \mathbb{R}^2$ such that

1) $\tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} \leq q_{i'} - \theta$

2) $\tilde{w}_{s_{j'}} > w_{s_{j'}}$ and $\tilde{w}_{b_{i'}} > w_{b_{i'}}$.

Then, we must have

$$\begin{aligned} c_{j'} + u_{i'} \leq w_{s_{j'}} + w_{b_{i'}} < \tilde{w}_{s_{j'}} + \tilde{w}_{b_{i'}} \leq q_{j'} - \theta \\ \Rightarrow \theta < q_{j'} - c_{j'} - u_{i'} \leq \max_j\{q_j - c_j\} - \min_i\{u_i\}. \end{aligned}$$

■

Theorem 4 states that stability is more easily attained if *PAM* is implemented. The intuition of the proof is similar to the one presented for theorem 2, with the difference that now, blocking coalitions will tend to hurt the platform even more, as it implies the transactions made by the coalitions are conducted outside of the platform, so they generate no revenue to the platform.

Example 4. ***(To guarantee stability, the platform may have incentives to implement PAM)*** *Consider a market where the set of sellers is given by $\{s_1, s_2, s_3, s_4\}$, and the set of buyers is given by $\{b_1, b_2, b_3, b_4\}$. Sellers' quality and costs are given by*

$$(q_1, q_2, q_3, q_4) = (20, 20, 20, 20) \quad \textit{and} \quad (c_1, c_2, c_3, c_4) = (2, 8, 14, 16),$$

respectively. Customers' outside option are given by

$$(u_1, u_2, u_3, u_4) = (0, 3, 5, 7).$$

If PAM is implemented, then

$$b_1 : s_1 \quad b_2 : s_2, \quad b_3 : s_3, \quad b_4 : s_4$$

If the platform charges commission fees $\psi = 0$ and $\tau = 1$, then this Pareto efficient match would generate a total of 3 transactions, resulting in a total profit of $3 \times \tau = 3$ to the platform.

But if it implemented the match

$$s_1 : b_4, \quad s_2 : b_3, \quad s_3 : b_2, \quad s_4 : b_1 \tag{29}$$

then 4 transactions would take place, so the platform's profits would equal to $4 \times \tau = 4$.

However, if $\theta = \tau$, it can be shown that the match 29 is unstable, as buyer b_1 would have incentives to form a blocking coalition with seller s_1, and buyer b_2 would have incentives to form a blocking coalition with seller s_3. Because these agents would transact outside of the platform, and because the remaining agents ended up *unmatched (as their tentative matches were clinched by the agents transacting outside of the platform), in the end no transactions would be made through the platform, causing the platform to earn zero profits.*

Meanwhile, $\tau = \theta$ implies that PAM is stable (theorem 4). So the platform may be better off implementing PAM and getting a profit of 3 *as opposed to implementing the match* (29) *and ending up with zero profits.*

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